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CHAPTER 1

INTRODUCTION

The finite element method[1] is a technique for obtaining approximate solutions to boundary value problems. To introduce the method, we present an elementary (some might say trivial) example of the deflection of a tightly stretched wire under a distributed load. This example is sufficient to (1) refresh the reader on some aspects of differential equations that will prove important for understanding approximate solutions, (2) introduce the concept of approximate solutions, and (3) actually define and illustrate the finite element method. This particular application was selected because the solution is easily visualized. However, any number of other applications could have been used, such as one-dimensional problems in heat transfer, porous flow, or electrostatics. The variables given here can be interpreted to correspond to one of these applications, and the reader is encouraged to do so if that is helpful.

1.1 GOVERNING EQUATION AND AN EXACT SOLUTION

Consider a tightly stretched wire as shown in Fig. 1.1. Under the proper circumstances, the deflection of the wire is accurately described by the solution of

$$T\frac{d^2y}{dx^2} + w(x) = 0.0 \tag{1.1}$$

where

T = tension in the wire $[F]$
y = deflection of the wire $[L]$[2]
w = distributed load $[F/L]$

The bracketed terms are the dimensions of the variables, where F represents force and L represents length.

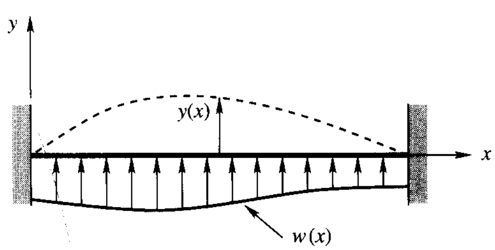


Figure 1.1. Deflection of a tightly stretched wire.

[1]To be abbreviated FEM throughout much of the text.
[2]Note that the symbol y represents a dependent variable and not a coordinate. Other symbols, such as u or v, could be used to eliminate any possible confusion; however, we have chosen to use y which conforms to the usage in many texts on ordinary differential equations. For two-dimensional problems in later chapters, y will indeed represent a coordinate.

The proper circumstances mentioned above are that T should be large enough and y small enough to ensure that T remains nearly constant during the deformation and that

$$\left(\frac{dy}{dx}\right)^2 \ll \left|\frac{dy}{dx}\right| \tag{1.2}$$

Before we obtain an exact solution, it is necessary to define a particular loading. The loading selected is one that does not complicate the mathematics beyond that needed to fulfill the purposes stated above, and is shown in Fig. 1.2.

The governing equation now becomes

$$\begin{aligned} T\frac{d^2y}{dx^2} - 3W &= 0 \quad \text{for} \quad 0 \le x \le L/2 \\ T\frac{d^2y}{dx^2} - W &= 0 \quad \text{for} \quad L/2 \le x \le L \end{aligned} \tag{1.3}$$

with boundary conditions

$$y(0) = y(L) = 0 \tag{1.4}$$

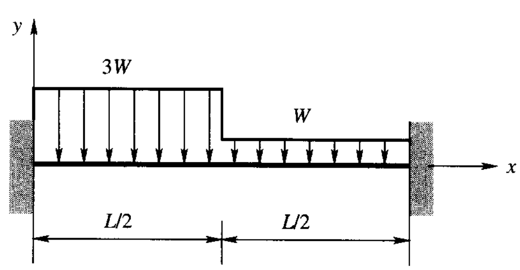


Figure 1.2. Loading on wire.

The loading is defined at all points but is discontinuous at $x = L/2$. The governing equation tells us that we should expect the same for the second derivative of y with respect to x. However, the first derivative must be continuous at $x = L/2$; otherwise the second derivative would not be defined at this point.

There are several approaches available to arrive at the exact solution. The one chosen here is to obtain the general solution for each of the two regions specified in Eq. 1.3, and require each one to have the same value for deflection and the same value for slope at $x = L/2$.

The general solutions for the two segments are

$$\begin{aligned} y(x) &= \frac{W}{T}\left(\frac{3}{2}x^2 + C_1 x + C_2\right) \quad \text{for} \quad 0 \le x \le L/2 \\ y(x) &= \frac{W}{T}\left(\frac{1}{2}x^2 + C_3 x + C_4\right) \quad \text{for} \quad L/2 \le x \le L \end{aligned} \tag{1.5}$$

The four constants of integration can be evaluated using the four conditions

$$\begin{aligned} y(x = 0) &= 0 \\ y(x = L) &= 0 \\ y\left(x = \frac{L}{2}-\right) &= y\left(x = \frac{L}{2}+\right) \\ \frac{dy}{dx}\left(x = \frac{L}{2}-\right) &= \frac{dy}{dx}\left(x = \frac{L}{2}+\right) \end{aligned} \tag{1.6}$$

Evaluation of these constants gives us

$$\begin{aligned} y(x) &= \frac{WL^2}{T}\left[\frac{3}{2}\left(\frac{x}{L}\right)^2 - \frac{5}{4}\left(\frac{x}{L}\right)\right] \quad \text{for} \quad 0 \le x \le L/2 \\ y(x) &= \frac{WL^2}{T}\left[\frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{4}\left(\frac{x}{L}\right) - \frac{1}{4}\right] \quad \text{for} \quad L/2 \le x \le L \end{aligned} \tag{1.7}$$

which is the exact solution to our governing equation. The plot of this function is shown in Fig. 1.3 at the end of the next section.

1.2 APPROXIMATIONS TO THE EXACT SOLUTION

We now consider how we would arrive at an approximate solution to our governing equation if the exact solution were not obtainable. Almost all methods for doing so use an approximating function with a finite number of degrees of freedom, such as

$$y(x) = (x)(x - L)\left[A_0 + A_1 x + A_2 x^2 + A_3 x^3\right] \tag{1.8}$$

or

$$y(x) = A_0 \sin\left(\frac{\pi x}{L}\right) + A_1 \sin\left(\frac{2\pi x}{L}\right) + A_2 \sin\left(\frac{3\pi x}{L}\right) \tag{1.9}$$

Here the degrees of freedom (the number of undetermined parameters) equal 4 in Eq. 1.8 and 3 in Eq. 1.9. Note that both approximations satisfy the boundary conditions independent of the parametric values. This is an important requirement for selecting approximating functions. Also worth noting is that the approximations can be made to converge to any continuous function by extending the series indicated. Such series are referred to as being *complete* or having the property of *completeness*, a topic covered in the mathematics of infinite series.

There are other series we could consider, including a finite element approximation. However, before we get to that, it is instructive to illustrate some of the methods used to obtain values for the parameters in these series. To do so, we will use the truncated polynomial series

$$y(x) = (x)(x - L)(A_0 + A_1 x) \tag{1.10}$$

Note that the dimensions of A_0 and A_1 must be $[L^{-1}]$ and $[L^{-2}]$, respectively. It is often best to write a series in a way that makes the parameters nondimensional, or at least in a way that gives them the same dimensions. However, to simplify the notation while retaining the physical dimensions of all quantities, the above form was chosen.

The exact solution satisfies the differential equation for all values of x. However, there are no values of A_0 and A_1 that would make our approximation capable of satisfying our equation at all points. This is easily seen by noting that the exact solution has a discontinuity in its second derivative, whereas all derivatives of our approximating function are continuous. Because there are no values for our parameters that will give us the exact solution, we must determine which values would be "best." This would not be a difficult task if we knew the exact solution. However, we are pretending not to know this, but only the differential equation that it must satisfy. We need, therefore, some technique for determining the best values for the parameters based only on the governing differential equation.

There are several ways to determine, or define, these "best" values, most of which are related to a function called the *residual*. This is the function obtained when the approximating function is substituted into the governing equation. For our case, this function is

$$\begin{aligned} R(x, A_0, A_1) &= 2TA_0 + T(6x - 2L)A_1 - 3W \quad \text{for} \quad 0 \le x \le L/2 \\ R(x, A_0, A_1) &= 2TA_0 + T(6x - 2L)A_1 - W \quad \text{for} \quad L/2 \le x \le L \end{aligned} \tag{1.11}$$

where we have indicated that the residual depends not only on x, but also on the values of the yet undetermined parameters A_0 and A_1. If it were possible to select values for A_0 and A_1 such that the residual would be zero for all values of x, then we would have the exact solution. However, as we have already noted, this is not possible. Hence, a compromise must be made. We now consider several well-known compromises.

1.2.1 Collocation. If we cannot make $R = 0$ at all points between 0 and L, we will make it zero at as many points as possible. Because we have only two undetermined parameters, we expect at most to be able to satisfy the equation at two points. We choose $x = L/4$ and $x = 3L/4$ as logical choices, and obtain

$$\begin{aligned} 2A_0 - \frac{L}{2}A_1 &= \frac{3W}{T} \\ 2A_0 + \frac{5L}{2}A_1 &= \frac{W}{T} \end{aligned} \tag{1.12}$$

which gives us

$$A_0 = \frac{4W}{3T}$$
$$A_1 = -\frac{2W}{3LT} \tag{1.13}$$

Hence, our approximate solution by collocation is

$$\frac{y}{L} = \frac{WL}{T}\left[-\frac{2}{3}\left(\frac{x}{L}\right)^3 + 2\left(\frac{x}{L}\right)^2 - \frac{4}{3}\left(\frac{x}{L}\right)\right] \tag{1.14}$$

Figure 1.3, at the end of this section, compares this solution with the exact solution as well as the other approximate solutions that follow. The error between it and the exact solution, is shown in Fig. 1.4. Note that the maximum error is approximately 0.4%.

Two things are worth noting. First, the results are good because our approximation was good—a cubic that was able to duplicate our exact solution very closely. That might not be true for a more complex loading. In such a case, we would need to add more terms (parameters) in our approximating function. Second, our solution matches the exact solution at $x = L/4$ and $x = 3L/4$ (not easily seen from the graph, but it does). This is not because these points were the collocation points; rather, it is more of an accident peculiar to this particular problem. In general this will not be the case. We satisfy the *differential equation* at collocation points, not the *solution*.

1.2.2 Least Squares. Rather than making $R(x, A_0, A_1) = 0$ at two points, we might want to minimize its magnitude in some average sense along the entire length from 0 to L. One way of doing this is to minimize

$$J = \int_0^L R^2\,dx \tag{1.15}$$

Clearly, J is greater than zero except for the case where R is everywhere zero, i.e., the exact solution. Therefore, of all possible functions that satisfy our boundary conditions, the exact solution produces the lowest value of J. We know, because our approximation does not contain the exact solution, that it cannot produce this minimum value; thus, we settle for as low a value of J as it is able to produce. That is, we find values of A_0 and A_1 that give J a stationary value. Thus, we solve

$$\frac{\partial J}{\partial A_0} = 0 \quad \text{and} \quad \frac{\partial J}{\partial A_1} = 0 \tag{1.16}$$

for A_0 and A_1.

Rather than integrate and then differentiate, it is easier to differentiate and then integrate; therefore, we write

$$\frac{\partial}{\partial A_i}\int_0^L R^2\,dx = \int_0^L 2R\frac{\partial R}{\partial A_i}\,dx = 0 \tag{1.17}$$

Equations 1.11 give us R, from which we obtain

$$\begin{aligned} \frac{\partial R}{\partial A_0} &= 2T \qquad \text{for} \quad 0 \le x \le L \\ \frac{\partial R}{\partial A_1} &= T(6x - 2L) \quad \text{for} \quad 0 \le x \le L \end{aligned} \tag{1.18}$$

From this we obtain the following two equations:

$$\begin{aligned} \frac{\partial J}{\partial A_0} &= \int_0^{L/2} [2T][2TA_0 + T(6x - 2L)A_1 - 3W]\,dx \\ &\quad + \int_{L/2}^{L} [2T][2TA_0 + T(6x - 2L)A_1 - W]\,dx = 0 \end{aligned} \tag{1.19}$$

$$\begin{aligned} \frac{\partial J}{\partial A_1} &= \int_0^{L/2} [6T(6x - 2L)][2TA_0 + T(6x - 2L)A_1 + 3W]\,dx \\ &\quad + \int_{L/2}^{L} [6T(6x - 2L)][2TA_0 + T(6x - 2L)A_1 + W]\,dx = 0 \end{aligned} \tag{1.20}$$

which gives, after integration and the solution of the resulting algebraic equations,

$$\begin{aligned} A_0 &= \frac{5W}{4T} \\ A_1 &= -\frac{W}{2LT} \end{aligned} \tag{1.21}$$

Hence, our approximate solution found by the least squares method is

$$\frac{y}{L} = \frac{WL}{T}\left[-\frac{1}{2}\left(\frac{x}{L}\right)^3 + \frac{7}{4}\left(\frac{x}{L}\right)^2 - \frac{5}{4}\left(\frac{x}{L}\right)\right] \tag{1.22}$$

The solution is shown in Fig. 1.3 and its error in Fig. 1.4. It has a slightly higher maximum error than the function found by collocation, but is still less than 1%.

1.2.3 Galerkin's Method. Another method for reducing the residual over the entire length is to make its weighted average zero with respect to as many independent weighting functions as there are undetermined parameters. That is,

$$\begin{aligned} \int_0^L W_0 R(x, A_0, A_1, \ldots, A_n)\,dx &= 0 \\ \int_0^L W_1 R(x, A_0, A_1, \ldots, A_n)\,dx &= 0 \\ &\vdots \\ \int_0^L W_n R(x, A_0, A_1, \ldots, A_n)\,dx &= 0 \end{aligned} \tag{1.23}$$

The weighting functions must be independent in order for the resulting algebraic equations to be independent. Galerkin's method uses the independent terms of the approximating function as the weighting functions; thus, there are always as many weighting functions as there are independent parameters. For our example, the method gives us

$$\int_0^{L/2}\left[x^2 - Lx\right]\left[2TA_0 + T(6x - 2L)A_1 + 3W\right]dx + \int_{L/2}^{L}\left[x^2 - Lx\right]\left[2TA_0 + T(6x - 2L)A_1 + W\right]dx = 0 \tag{1.24}$$

$$\int_0^{L/2}\left[x^3 - Lx^2\right]\left[2TA_0 + T(6x - 2L)A_1 + 3W\right]dx + \int_{L/2}^{L}\left[x^3 - Lx^2\right]\left[2TA_0 + T(6x - 2L)A_1 + W\right]dx = 0 \tag{1.25}$$

Integration of these equations produces two algebraic equations in A_0 and A_1, the solution of which gives

$$A_0 = \frac{21W}{16T} \qquad A_1 = -\frac{5W}{8TL} \tag{1.26}$$

Thus, the approximation solution by Galerkin's method is

$$\left(\frac{y}{L}\right) = \frac{WL}{T}\left[-\frac{5}{8}\left(\frac{x}{L}\right)^3 + \frac{31}{16}\left(\frac{x}{L}\right)^2 - \frac{21}{16}\left(\frac{x}{L}\right)\right] \tag{1.27}$$

1.2.4 The Ritz Method. For our tight wire problem it is possible to write an expression for the total potential energy of the system as a function of the deflection, $y(x)$. We know from the principles of mechanics that this scalar function attains a minimum value corresponding to the equilibrium position of the wire. That is, if the potential energy of the system is calculated using the exact solution of the governing differential equation, its value will be lower than that calculated when any other deflected shape is used for the calculation. Thus, as we did for the least squares method, we will judge the best values for A_0 and A_1 as those that give the smallest potential energy to the system, i.e., those values that produce a potential energy nearest to the potential energy that the exact solution has.

The potential energy of the system is given by

$$E = \int_0^L\left[\frac{1}{2}T\left(\frac{dy}{dx}\right)^2 - wy\right]dx \tag{1.28}$$

for any loading $w = w(x)$ and any deflection $y = y(x)$.[3] Similar expressions exist for other applications.

[3]This can be shown using elementary mechanics. The first term in the integrand represents the potential energy of the distributed load, and the second term in the integrand represents the increase in potential energy (strain energy) in the wire due to the deflection.

We will shortly show that the minimization of such an expression is equivalent to satisfying the governing differential equation.

When an approximating function is substituted into the above integral, E becomes a function of the undetermined parameters used for the function; thus, to obtain a stationary value, we enforce

$$\begin{aligned} \frac{\partial E}{\partial A_i} &= \frac{\partial}{\partial A_i}\int_0^L \left[\frac{1}{2}T(y')^2 - wy\right] dx \\ &= \int_0^L \frac{\partial}{\partial A_i}\left[\frac{1}{2}T(y')^2 - wy\right] dx \\ &= \int_0^L \left[y'T\frac{\partial(y')}{\partial A_i} - w\frac{\partial(y)}{\partial A_i}\right] dx \end{aligned} \tag{1.29}$$

For our two-parameter approximating function,

$$\begin{aligned} y &= (x^2 - xL)A_0 + (x^3 - x^2L)A_1 \\ y' &= (2x - L)A_0 + (3x^2 - 2xL)A_1 \\ \frac{\partial(y)}{\partial A_0} &= (x^2 - xL) \\ \frac{\partial(y)}{\partial A_1} &= (x^3 - x^2L) \\ \frac{\partial(y')}{\partial A_0} &= (2x - L) \\ \frac{\partial(y')}{\partial A_1} &= (3x^2 - 2xL) \\ w &= \begin{cases} -3W & \text{for} \quad 0 \le x \le L/2 \\ -W & \text{for} \quad L/2 \le x \le L \end{cases} \end{aligned} \tag{1.30}$$

Substitution of these functions into Eq. 1.29 creates, as with the previous methods, two linear algebraic equations for A_0 and A_1. Solving, we find that the resulting values of our two parameters are identical to the ones we obtained by Galerkin's method. There is a reason for this, which will soon be explained. The plots of this function and its error are the same as the plots for Galerkin's method in Figs. 1.3 and 1.4.

For many of the boundary value problems associated with engineering analysis, there is a corresponding scalar function such as we have used here. These scalar functions of functions[4] are referred to as *functionals*. The procedures for determining the functional corresponding to a given differential equation, or vice versa, are the subject of the calculus of variations. The principle that the solution of a given boundary value problem

[4]The function for E was a function of the function $y(x)$.

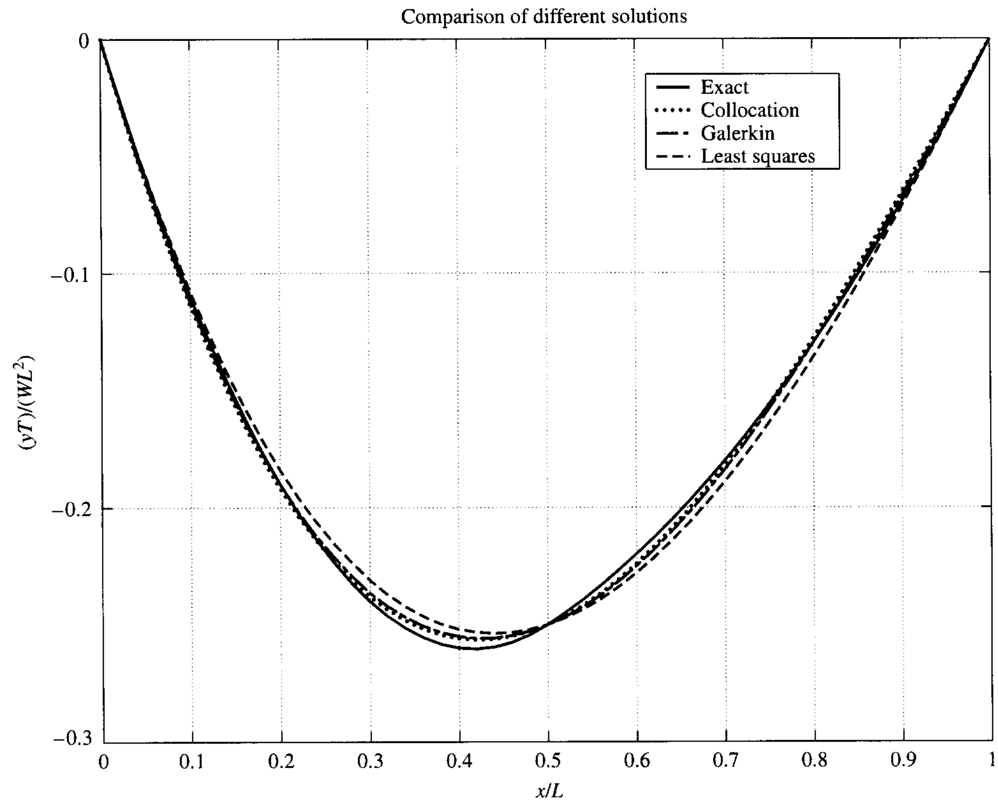


Figure 1.3. Comparison of the approximate solutions to the exact solution.

corresponds to a stationary value of a functional is referred to as a *variational principle*. Finally, the concept of using a variational principle to judge the quality of an approximate solution is referred to as the Ritz method. We will spend considerable time exploring these ideas because they represent the foundations of the finite element method.

1.2.5 Results. In comparing the preceding methods, it is important not to place emphasis on which one has the smallest error. The errors shown depend more on the particular problem and the approximating function selected than on the method. All of the methods converge to the exact solution as the number of terms used in the approximation increase, provided that the series is complete. The question of convergence is important and will be discussed at the conclusion of this chapter.

For now, however, let us see how each of the methods fared. Figure 1.3 illustrates the deflection found for each method (the Ritz method is the same as Galerkin's). Figure 1.4 illustrates the difference between the exact solution and the approximate solutions. Again, you should understand that the closeness of these approximations to the exact solution is due to the fact that our approximating function, with only two independent parameters, can be made very close to the piecewise parabola that is the exact solution. In general, many more terms are necessary to obtain the accuracy needed in most engineering analyses.

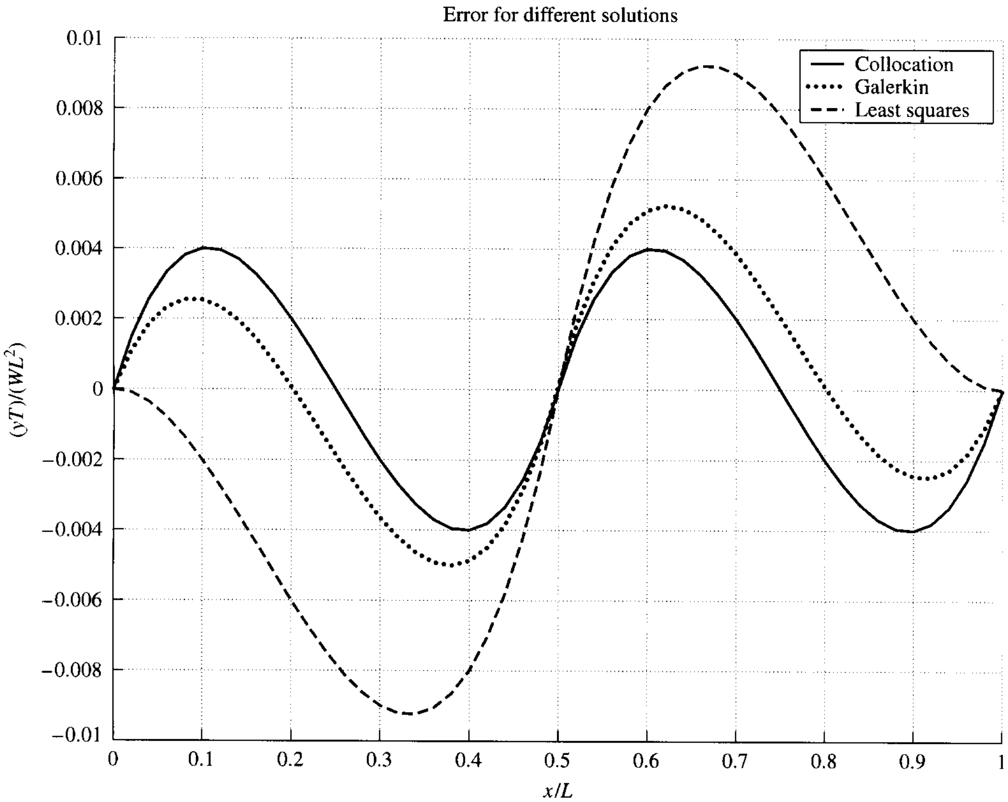


Figure 1.4. Difference between the exact solution and the approximate solutions along the wire.

1.3 FINITE ELEMENT APPROXIMATIONS

There is a very important difference between the Ritz method and the other three methods illustrated in the previous section, and that difference allows us to introduce finite element approximations. Whereas the first three methods required that the second derivative of the approximating function exist, the Ritz method required only the existence of the first derivative. This weaker demand allows us to use piecewise linear functions as our approximating function. Such approximations are the foundations of the finite element method, several of which are illustrated in Fig. 1.5. You should note three characteristics of these functions, which we will discuss in much more detail in later chapters:

1. The functions are continuous, although their first derivatives (slopes) are not.
2. Each function is completely defined by its nodal point values; hence, these values serve as the undetermined parameters of the approximating function.
3. The segments, or elements, need not be of equal length.

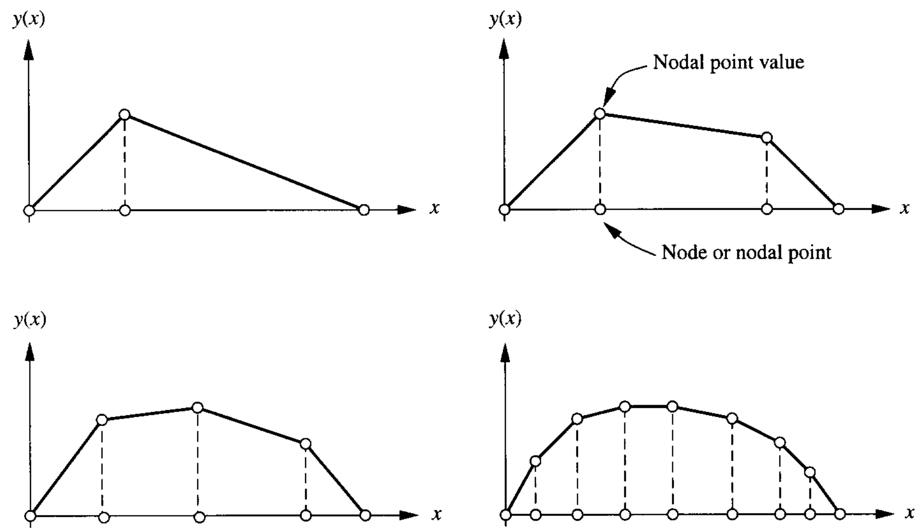


Figure 1.5. Finite element approximations.

Let us now use one such function to approximate the deflection of our tight wire. We choose the particular one illustrated in Fig. 1.6, given by

$$y(x) = \begin{cases} (3Y_2/L)x & \text{for } 0 \le x \le L/3 \\ (2Y_2 - Y_3) + (3/L)(Y_3 - Y_2)x & \text{for } L/3 \le x \le 2L/3 \\ 3Y_3 - (3Y_3/L)x & \text{for } 2L/3 \le x \le L \end{cases} \tag{1.31}$$

The energy functional,

$$E = \int_0^L \left[-wy + 0.5T\left(\frac{dy}{dx}\right)^2 \right] dx \tag{1.32}$$

requires the use of the derivative of this function, which is

$$\frac{dy}{dx} = \begin{cases} 3Y_2/L & \text{for } 0 \le x \le L/3 \\ (3/L)(Y_3 - Y_2) & \text{for } L/3 \le x \le 2L/3 \\ -3Y_3/L & \text{for } 2L/3 \le x \le L \end{cases} \tag{1.33}$$

Substitution of the function, its derivative, and the loading function into Eq. 1.32 gives us the potential energy as a function of the deflections, Y_2 and Y_3, which we indicate by writing

$$E = E(Y_2, Y_3) \tag{1.34}$$

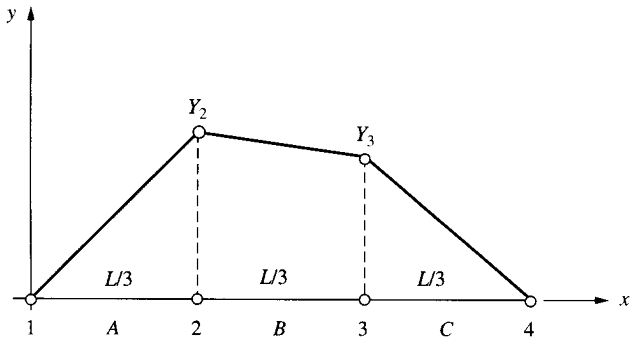


Figure 1.6. Finite element approximation to the deflection of the tight wire.

The Ritz method states that the best approximation to the exact solution that these two parameters can give are the two values that create the lowest value for E, that is, the value of E that is closest to the value obtained by the exact solution. Thus, we seek the two nodal values that will satisfy

$$\frac{\partial E}{\partial Y_2} = 0 \quad \text{and} \quad \frac{\partial E}{\partial Y_3} = 0 \tag{1.35}$$

As before, it would be easier to differentiate and then integrate, but this time we choose to integrate first and obtain the potential energy in terms of our two nodal values for displacement. This approach emphasizes that the potential energy is a quadratic function of the displacements; hence, it has one, and only one, stationary value.

A significant consequence of piecewise approximations is that the integration can be performed element by element and summed to obtain the total integral. This is the approach we now use.

The contribution to E from element A is

$$E_A = -\int_0^{L/3} \left[-3W\left[(3Y_2/L)x\right] \right] dx + \int_0^{L/3} \left[\frac{1}{2} T \left[3Y_2/L\right]^2 \right] dx \tag{1.36}$$

$$E_A = \left[\frac{WL}{2}\right] Y_2 + \left[\frac{3T}{2L}\right] Y_2^2 \tag{1.37}$$

The contribution to E from element B is

$$\begin{aligned} E_B = &-\int_{L/3}^{L/2} \left[-3W\left[(2Y_2 - Y_3) + (3/L)(Y_3 - Y_2)x\right] \right] dx \\ &-\int_{L/2}^{2L/3} \left[-W\left[(2Y_2 - Y_3) + (3/L)(Y_3 - Y_2)x\right] \right] dx \\ &+\int_{L/3}^{2L/3} \left[\frac{1}{2} T \left[(3/L)(Y_3 - Y_2)\right]^2 \right] dx \end{aligned} \tag{1.38}$$

$$E_B = \left[\frac{5WL}{12}\right]Y_2 + \left[\frac{3WL}{12}\right]Y_3 + \left[\frac{3T}{2L}\right](Y_3 - Y_2)^2 \tag{1.39}$$

The contribution to E from element C is

$$E_C = -\int_{2L/3}^{L}\left[-W\left[3Y_3 - (3Y_3/L)x\right]\right]dx + \int_0^{L/3}\left[\frac{1}{2}T\left[-3Y_3/L\right]^2\right]dx \tag{1.40}$$

$$E_C = \left[\frac{WL}{6}\right]Y_3 + \left[\frac{3T}{2L}\right]Y_3^2 \tag{1.41}$$

The total energy is

$$E = E_A + E_B + E_C \tag{1.42}$$

$$E = \left[\frac{11WL}{12}\right]Y_2 + \left[\frac{5WL}{12}\right]Y_3 + \left(\frac{3T}{L}\right)\left(Y_2^2 - Y_2Y_3 + Y_3^2\right) \tag{1.43}$$

which is a quadratic function of Y_2 and Y_3. It thus has a single, stationary value where

$$\frac{\partial E}{\partial Y_2} = 0.0 \tag{1.44}$$

and

$$\frac{\partial E}{\partial Y_3} = 0.0 \tag{1.45}$$

Thus, we obtain

$$\begin{aligned}\left(\frac{11WL}{12}\right) + \left(\frac{6T}{L}\right)Y_2 - \left(\frac{3T}{L}\right)Y_3 &= 0.0\\ \left(\frac{5WL}{12}\right) - \left(\frac{3T}{L}\right)Y_2 + \left(\frac{6T}{L}\right)Y_3 &= 0.0\end{aligned} \tag{1.46}$$

or, in matrix notation,

$$\begin{bmatrix}\left(\frac{6T}{L}\right) & \left(\frac{-3T}{L}\right)\\ \left(\frac{-3T}{L}\right) & \left(\frac{6T}{L}\right)\end{bmatrix}\begin{Bmatrix}Y_2\\ Y_3\end{Bmatrix} = \begin{Bmatrix}-\frac{11WL}{12}\\ -\frac{5WL}{12}\end{Bmatrix} \tag{1.47}$$

The solution is

$$\begin{Bmatrix} Y_2 \\ Y_3 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{WL^2}{4T} \\ -\dfrac{7WL^2}{36T} \end{Bmatrix} \tag{1.48}$$

This solution is shown in Fig. 1.7. The nodal point values in this case turn out to coincide with the exact solution. However, this will not happen under different circumstances (i.e., a different governing equation).[5]

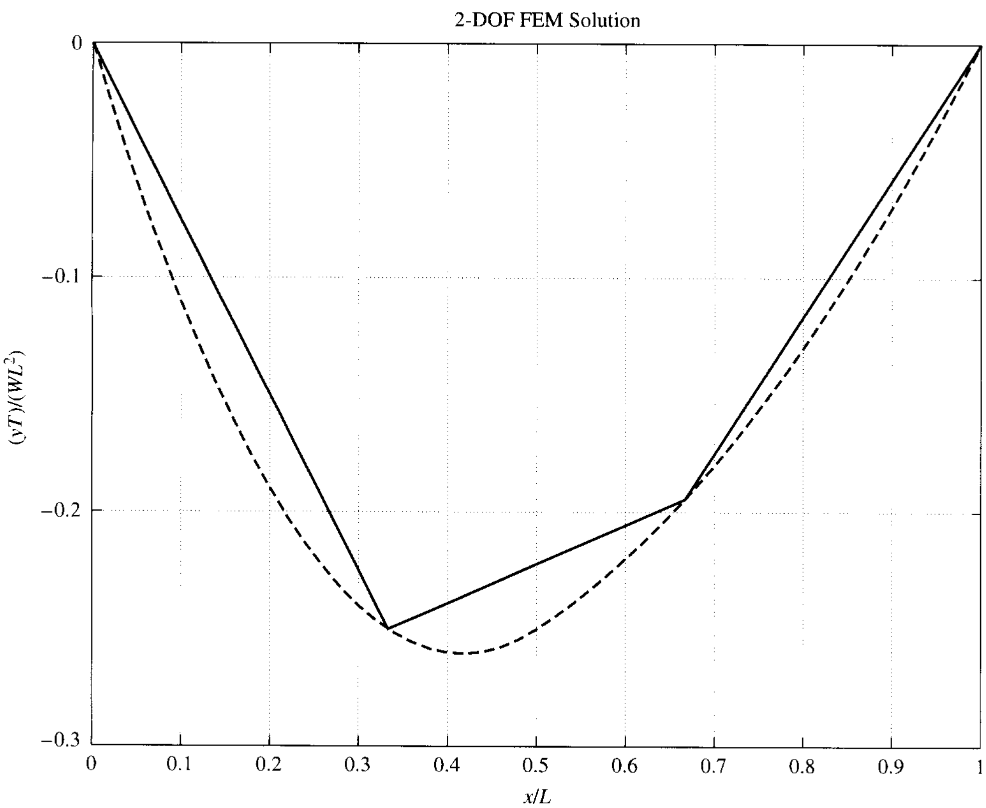


Figure 1.7. Finite element solution.

[5]The deflection of a tightly stretched wire under the action of concentrated loads is made up of piecewise linear functions, the same as our finite element approximation. Hence, our finite element approximations can be made to correspond to the exact solution simply by placing nodes at each point where there is a concentrated load. Furthermore, it can be shown that the exact solution for the deflection of a wire at any two points is the same for all statically equivalent loadings between these two points. These two facts are the reason for the remarkable results here. Proof of the latter of the two statements is an interesting exercise for students of applied mechanics.

You have probably noted that this solution does not compare well with those found by our two-term polynomial approximation. However, as more terms are added, the polynomial and trigonometric approximations become ever more difficult to evaluate, whereas the finite element solution simply repeats the same calculations, over and over, for each element. This advantage becomes pronounced for partial differential equations with irregular boundaries and nonconstant parameters.

1.4 BASIS FUNCTIONS

We have referred to our approximating functions as series. This is, of course, obvious for the polynomial and trigonometric approximations; however, finite element approximations do not, at first glance, appear to be a truncated series. Nevertheless, in much of the literature associated with the finite element method the approximation is treated as a series and written in the form

$$y(x) = \sum M_i(x) Y_i \tag{1.49}$$

When this is done, the functions $M_i(x)$ are called basis functions, with one such function associated with each node. A set of these functions for a four-node approximation is shown in Fig. 1.8.

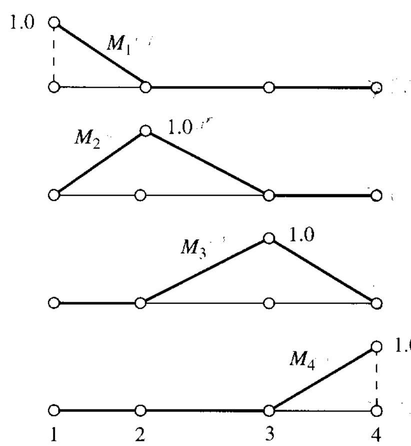


Figure 1.8. Basis functions for a four-node approximation.

Note that the magnitude of each function at its respective node is unity and that it is zero at all other nodes. Hence, their sum would have a value of unity at each node, and the function would simply be the straight line $y = 1.0$. If each basis function is multiplied by a constant A_i, then the sum of these new functions would be a piecewise linear function having a value at each node equal to the corresponding value of A_i for that node. As an example, if

$$A_1 = 0, \qquad A_2 = 4, \qquad A_3 = 3, \qquad A_4 = -1$$

then

$$y(x) = \sum A_i M_i(x) \tag{1.50}$$

is the function shown in Fig. 1.9.

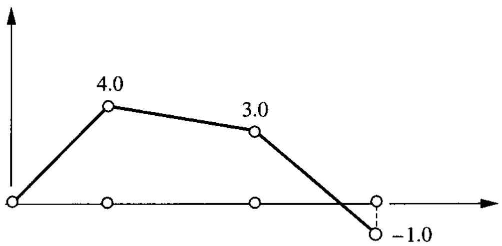


Figure 1.9. $\sum M_i Y_i$ for $\{Y\} = \{0, 4, 3, -1\}$.

1.5 WEAK FORMULATION OF PROBLEM

As we have seen, the use of a piecewise linear approximation depends on being able to express our problem in terms of the potential energy functional, Eq. 1.28. We also saw that for our polynomial approximation, the Ritz method produced the same results as Galerkin's method, and we noted that this was no accident. We now show how to obtain the integral formulation from the differential equation—that is, how to obtain the weaker formulation that requires only the existence of the first derivative from the stronger formulation that requires the existence of the second derivative. Both formulations are valid, and the solution to one is the solution to the other. Note specifically that what follows is not directly related to approximation theory, although we will certainly use the results for that purpose. That is, the original differential equation for our tight wire problem and the expression for the potential energy of the wire are physical statements of the physics of the problem and have no direct correlation with approximation theory.

Consider the original differential equation for the deflection of the tight wire,

$$T\frac{d^2y}{dx^2} + w(x) = 0 \tag{1.51}$$

Because this equation must be satisfied at all points (i.e., must be zero at all points),

$$B(x)\left[T\frac{d^2y}{dx^2} + w(x)\right] = 0 \tag{1.52}$$

must also be satisfied (i.e., equal to zero) for any function $B(x)$. If so, then its integral must likewise equal zero,

$$\int_0^L B(x)\left[T\frac{d^2y}{dx^2} + w(x)\right] dx = 0 \tag{1.53}$$

Interpret this equation as a test that the trial function, $y(x)$, must pass for all permissible test functions, $B(x)$, if it is the exact solution.

We now assume that the test functions, $B(x)$, are sufficiently well behaved that the integrand remains integrable. Thus, we integrate by parts to obtain

$$\int_0^L \left[\frac{d}{dx}\left(BT\frac{dy}{dx}\right) - \frac{dB}{dx}T\frac{dy}{dx} + Bw \right] dx = 0 \tag{1.54}$$

$$BT\left.\frac{dy}{dx}\right]_0^L - \int_0^L \left(\frac{dB}{dx}T\frac{dy}{dx} - Bw\right) dx = 0 \tag{1.55}$$

The integrated first term, in later problems, will allow us to specify boundary conditions other than what we have. For now, however, we assume the original boundary conditions of our tight wire problem, i.e., $y(x)$ known at both $x = 0$ and $x = L$, and let Eq. 1.55 apply only to functions that meet these conditions. This being the case, there is no need to test $y(x)$ at these points, and we restrict our test functions to those that are zero at $x = 0$ and $x = L$. Thus, for our specific problem,

$$\int_0^L \left(\frac{dB}{dx}T\frac{dy}{dx} - Bw\right) dx = 0.0 \tag{1.56}$$

Note that Eq. 1.56 applies to the $y(x)$ that satisfies our boundary conditions at $x = 0$ and $x = L$, whether $y(x)$ is specified as zero or some other value at these points. On the other hand, the test functions $B(x)$ must be zero at these points; otherwise we would have to include the first term in Eq. 1.55, which requires the value of dy/dx. This we ordinarily do not know when y is known. More on this point in the next chapter.

We now have two ways of defining our problem:

1. The deflection of the tight wire, $y(x)$, must satisfy the specified boundary conditions at $x = 0$ and $x = L$, and

$$T\frac{d^2y}{dx^2} + w(x) = 0$$

at every point $0 < x < L$.

2. The deflection of the tight wire, $y(x)$, must satisfy the specified boundary conditions at $x = 0$ and $x = L$, and

$$\int_0^L \left(\frac{dB}{dx}T\frac{dy}{dx} - Bw\right) dx = 0.0$$

for all test functions $B(x)$ for which $B(0) = B(L) = 0$.

The first formulation places stronger demands on the smoothness of the function than does the second formulation; hence, the designations *strong form* and *weak form* are used for the first and second forms, respectively. It will be this second formulation, i.e., the weak formulation, that will be the starting point of all our finite element formulations.

In order to see the equivalence of the above weak form to the potential energy functional used for the Ritz method, we must go back to that formulation, but this time differentiate and then integrate our equation.

$$
\begin{aligned}
\frac{\partial E}{\partial A_1} &= \frac{\partial}{\partial A_i}\int_0^L \left[\frac{1}{2}T\left(\frac{dy}{dx}\right)^2 + wy\right] dx \\
&= \int_0^L \left[T\frac{\partial}{\partial A_i}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + w\left(\frac{\partial y}{\partial A_i}\right)\right] dx \qquad (1.57) \\
&= \int_0^L \left[\frac{d}{dx}\left(\frac{\partial y}{\partial A_i}\right)T\left(\frac{dy}{dx}\right) + w\left(\frac{\partial y}{\partial A_i}\right)\right] dx = 0
\end{aligned}
$$

Comparison of this final form of the Ritz method with Eq. 1.56 shows that they are identical if we use as our test functions

$$B = \frac{\partial y}{\partial A_i}$$

Likewise, comparison of Galerkin's formulation (Eqs. 1.23) with Eq. 1.53 shows that they are identical if we, again, use as our test functions

$$B = \frac{\partial y}{\partial A_i}$$

Therefore, both Galerkin's method and the Ritz method simply represent use of the weak form of our problem to evaluate the undetermined parameters of an approximate solution.

1.6 SOME CONCLUDING REMARKS

We have assumed that our approximating functions are complete—that as we add more terms, the series can be made as close to the exact solution as we desire. We accepted that the polynomial and trigonometric series are complete. By the very nature of our finite element approximation, it is complete. However, even if we are confident that our approximating function can be made close to the exact solution, how do we know that the methods we have used to determine the parametric value actually force the approximation to be close to the exact solution? In general, this is a difficult question to answer and is best left to texts on that subject. However, it is worthwhile to develop an intuitive feel for why these methods work. A basic characteristic of all four of the methods is that they create a set of linear algebraic equations for the undetermined parameters. This will be true whenever these methods are used to approximate solutions of linear differential equations. Thus, in what follows, keep in mind that each method can produce only one set of values. Let us now review the four methods we introduced in this chapter.

Collocation. Of the methods presented, collocation is perhaps the most believable. As we increase the number of terms in our approximating function, we are able to increase the number of points at which the approximation exactly satisfies the differential equation. Hence, in the limit, among all the functions we are considering, we can both find a function that can duplicate the exact solution and find a function that satisfies the differential equation everywhere. Because there can be only one set of parameters for the function that satisfies the differential equation everywhere, and the exact solution is a function that does the same, the approximating function should approach the exact solution.

Least Squares. Here we seek to find the parametric values of our approximating function that minimize

$$L = \int_0^L R^2\, dx$$

Because our approximating function can be forced, in the limit, to approach any function, and because the exact function minimizes L, we conclude that by forcing the parameters to minimize L, the approximation must be forced to approach the exact solution.

Galerkin's Method. This is perhaps the most obscure of the methods we studied. What it requires is that our set of parametric values force the following equations to be satisfied:

$$\int_0^L P_1(x)R(x)\, dx = 0$$

$$\int_0^L P_2(x)R(x)\, dx = 0$$

$$\vdots$$

$$\int_0^L P_n(x)R(x)\, dx = 0$$

Here $R(x)$ is the residual and $P_i(x)$ are the independent terms in the approximating function

$$y(x) = \sum_{i=1}^{n} A_i P_i(x)$$

Because each equation is equal to zero, we can multiply each by an arbitrary constant and then sum them; thus

$$\int_0^L B_1 P_1(x)R(x)\, dx = 0$$

$$\int_0^L B_2 P_2(x)R(x)\, dx = 0$$

$$\vdots$$

$$\int_0^L B_n P_n(x)R(x)\, dx = 0$$

and

$$\sum_{i=1}^{n} \int_0^L B_i P_i(x)R(x)\, dx = 0$$

$$\int_0^L \sum_{i=1}^{n} B_i P_i(x)R(x)\, dx = 0$$

$$\int_0^L B(x)R\, dx = 0$$

where

$$B(x) = \sum_{i=1}^{n} B_i P_i(x)$$

Because the parameters are arbitrary and because the series is complete, we can make $B(x)$ approach any function we desire, including $R(x)$. This requires that, in the limit,

$$\int_0^L R^2 \, dx \to 0$$

Therefore, in the limit $R \to 0$; thus, our approximating function must approach the exact solution.

The Ritz Method. This is similar to the least squares method in that we force a functional to have a minimum value. In our case the functional was

$$E = \int_0^L \left[\frac{1}{2} T \left(\frac{dy}{dx} \right)^2 + wy \right]$$

Because our approximating function can be forced, in the limit, to approach any function, and because the exact function minimizes E, then forcing the parameters to minimize E should likewise force the approximation to approach the exact solution.

The preceding arguments do not constitute rigorous proofs for convergence, but they do point to how such proofs are created and, therefore, provide insight as to how these methods work. Such insight will be helpful in your understanding the material to follow as well as in your later use of the finite element method in whatever capacity you choose.

EXERCISES

Study Problems

S1. Consider another application associated with the example problem, such as heat conduction, porous flow, or electrostatics, and identify the physical significance of T and W in the governing equation.

S2. For each of the methods, use only one degree of freedom in the polynomial series, e.g.,

$$y(x) = (x)(x - L)A_0$$

and determine the "best" value for A_0 that each method gives. Use the loading shown in Fig. 1.2.

S3. For each of the methods, use the approximating function

$$y(x) = A_0 \sin\left(\frac{\pi x}{L}\right)$$

and determine the "best" value for A_0 that each method gives. Use the loading shown in Fig. 1.2.

S4. For the loading shown in Fig. 1.2, determine the potential energy of the tight wire problem corresponding to the exact solution. Give your answer in terms of W^2L^3/T.

S5. For the loading shown in Fig. 1.2, determine the potential energy that the two-parameter approximate solutions give to the tight wire problem. Give your answer in terms of W^2L^3/T.

S6. Consider the loading on the wire constant over its entire length and derive the exact solution. Show that all four methods give the exact solution with the approximating function

$$y(x) = (x)(x - L)A_0$$

S7. Fully explain why the finite element series is complete. What qualifying statement would you have to make concerning how the series is made?

Numerical Experiments and Code Development

N1. For the tight wire problem with a constant uniform load equal to W, and $T = 100WL$, calculate the potential energy as E/WL^2 using

$$y(x) = A_0 \sin\left(\frac{\pi x}{L}\right)$$

and plot it as a function of A_0/L. On the same plot, indicate the potential energy that the exact solution gives.

N2. For the tight wire problem with a constant uniform load equal to W, calculate

$$J = \int_0^L R^2\, dX$$

using

$$y(x) = A_0 \sin\left(\frac{\pi x}{L}\right)$$

and plot J/W^2L as a function of A_0/L. Let $T = 100WL$. What is the value of J that the exact solution gives?

N3. For the tight wire problem with a constant load equal to W, use the approximating function

$$y = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{L}\right)$$

and write a computer program to calculate the values of A_n by collocation for any N. Plot $(y_{\text{exact}} - y_{\text{approx}})$ as a function of x/L for $n = 1, 2, 3,$ and 4.

CHAPTER 2

CALCULUS OF VARIATIONS

In this chapter we set forth some of the basic concepts of the calculus of variations that we will use in developing the finite element method. In the previous chapter we defined the potential energy associated with the tight wire in terms of its deflection $y(x)$. The potential energy, therefore, is a scalar function of the function $y(x)$ and is referred to as a functional. If the deflection of the wire is changed a small amount, the potential energy will likewise change. The relationship between a change in a function $y(x)$ and the corresponding change in its dependent functional is the subject matter of the calculus of variations. It has a marked similarity to differential calculus, where one wishes to determine the relationship between a change in a scalar x and the differential of the function $y(x)$, i.e., $dy = (dy/dx)\,dx$.

2.1 THREE FUNDAMENTAL RELATIONSHIPS

We begin by defining the notation used for a change (or variation) in y as $\delta y = \epsilon\eta(x)$. Here η is a function of x, and ϵ is a scalar that we use to control the magnitude of the variation. Thus,

$$y + \delta y = y + \epsilon\eta(x) \tag{2.1}$$

These functions can be visualized as shown in Fig. 2.1.

In the following, we let $y(x)$ represent the (unknown) solution to a specific problem and $\epsilon\eta(x)$ an arbitrary variation of this solution. Usually we will study only small variations of $y(x)$, such that

$$\epsilon^2 \ll |\epsilon| \tag{2.2}$$

We will follow the practice that the symbol δy means that the variation is small in the sense of Eq. 2.2.

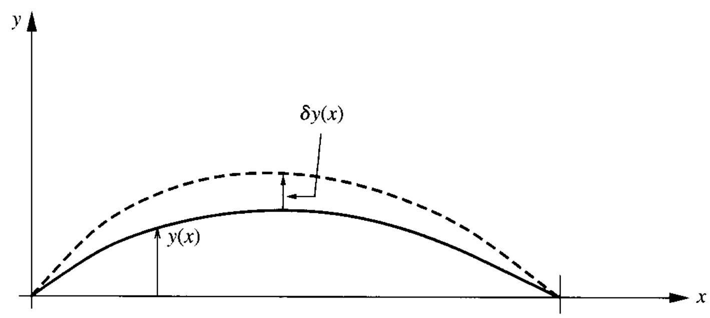


Figure 2.1. Variation of the wire deflection.

Now let us look at some important relationships. We first consider the derivative of $y(x)$ and its variation due to a variation of $y(x)$. By definition, the variation of the derivative is

$$\delta \frac{dy}{dx} = \frac{d(y + \delta y)}{dx} - \frac{dy}{dx} \tag{2.3}$$

which, when expanded, gives us

$$\delta \frac{dy}{dx} = \frac{dy}{dx} + \frac{d(\delta y)}{dx} - \frac{dy}{dx} \tag{2.4}$$

Therefore,

$$\delta \frac{dy}{dx} = \frac{d}{dx}\delta y \tag{2.5}$$

That is, the variation of the derivative is equal to the derivative of the variation. We will use the interchangeability of these operations a great deal in developing our finite element equations.

Next we consider the integral of $y(x)$ and its variation due to a variation of $y(x)$. By definition, we have

$$\delta\left[\int y(x)\,dx\right] = \left[\int [y(x) + \delta y(x)]\,dx\right] - \left[\int [y(x)]\,dx\right] \tag{2.6}$$

Upon expanding the second integral, we obtain

$$\delta\left[\int y\,dx\right] = \left[\int y\,dx\right] + \left[\int \delta y\,dx\right] - \left[\int y\,dx\right] \tag{2.7}$$

which is simply

$$\delta \int y\,dx = \int \delta y\,dx \tag{2.8}$$

That is, the variation of the integral is equal to the integral of the variation. Thus, the operations of integration and variation are interchangeable.

Now we consider a function $F(y)$, where $y = y(x)$, and pose the question, how does the function F change when $y(x)$ is changed from one function to another? That is, suppose the function

$$y(x) = Y(x) \tag{2.9}$$

is changed to the function

$$y(x) = Y(x) + \epsilon\eta(x) \tag{2.10}$$

Then the change in $F(y(x))$ would be

$$\delta F(y) = F\left[Y(x) + \epsilon\eta(x)\right] - F\left[Y(x)\right] \tag{2.11}$$

The first term on the right-hand side can be expanded in a Taylor's series about $\epsilon = 0$ to obtain

$$
\begin{aligned}
F(y) &= F\left[Y(x) + \epsilon\eta(x)\right] \\
&= F\Big|_{y=Y} + \frac{dF}{d\epsilon}\Big|_{y=Y}\epsilon + \frac{1}{2}\frac{d^2F}{d\epsilon^2}\Big|_{y=Y}\epsilon^2 + O(\epsilon^3)
\end{aligned}
\tag{2.12}
$$

For small ϵ, we may neglect the higher-order terms in ϵ and write

$$\delta F = F(Y) + \frac{dF(Y)}{d\epsilon}\epsilon - F(Y) \tag{2.13}$$

or simply

$$\delta F = \frac{dF}{d\epsilon}\epsilon \tag{2.14}$$

The derivative can be written as

$$\frac{dF}{d\epsilon} = \frac{dF}{dy}\frac{dy}{d\epsilon} = \frac{dF}{dy}\eta \tag{2.15}$$

where we have used

$$y = Y + \epsilon\eta \tag{2.16}$$

Hence,

$$\delta F = \left(\frac{dF}{dy}\right)\eta\epsilon \tag{2.17}$$

Because $\eta\epsilon$ represents the small change of the argument y (see Eq. 2.1) of the function $F(y)$, we can write

$$\delta F = \left(\frac{dF}{dy}\right)\delta y \tag{2.18}$$

We have thus obtained an expression for the infinitesimal variation of the function $F(y)$ due to an infinitesimal change in the function $y(x)$, which is completely analogous to the expression for a differential that we have from differential calculus.

2.2 APPLICATION TO THE TIGHT WIRE PROBLEM

We now apply the result of the previous section to our tight wire problem. In Chapter 1, two formulations were given for the solution of this problem: the strong form, which stated that the deflection must satisfy

$$T\frac{d^2y}{dx^2} + w(x) = 0.0 \tag{2.19}$$

at every point, and the weak form, which stated that the deflection must give a stationary value to the functional

$$E(y) = \int_0^L \left[\frac{1}{2}T\left(\frac{dy}{dx}\right)^2 - wy\right]dx \tag{2.20}$$

We now demonstrate the equivalence of these two formulations by showing

1. If $y(x)$ satisfies Eq. 2.19, then when it is substituted into Eq. 2.20 and given a small variation $\delta y(x)$, there will be no variation of $E(y)$, i.e., $\delta E = 0$.
2. If $y(x)$ gives a stationary value to $E(y)$, i.e., $\delta E = 0$ for any δy, then $y(x)$ satisfies Eq. 2.19.

We begin by taking the variation of E with respect to y.

$$\delta E(y) = \delta \int_0^L \left[\frac{1}{2}T\left(\frac{dy}{dx}\right)^2 - wy\right]dx \tag{2.21}$$

Interchanging the operations of variation and integration, we obtain

$$\delta E(y) = \int_0^L \delta\left[\frac{1}{2}T\left(\frac{dy}{dx}\right)^2 - wy\right]dx \tag{2.22}$$

Application of the standard formula of differential calculus gives us

$$\delta E(y) = \int_0^L \left[T\left(\frac{dy}{dx}\right)\delta\left(\frac{dy}{dx}\right) - w\,\delta y\right]dx \tag{2.23}$$

We now interchange the operations of variation and differentiation to obtain

$$\delta E(y) = \int_0^L \left[T\left(\frac{dy}{dx}\right)\frac{d}{dx}(\delta y) - w\,\delta y\right]dx \tag{2.24}$$

To arrive at this point we used all three relationships developed in the previous section. From here on, however, we need only to follow the rules of differential and integral calculus. We next integrate Eq. 2.24 by parts; that is, we use

$$\frac{d(AB)}{dx} = \frac{dA}{dx}B + \frac{dB}{dx}A \tag{2.25}$$

to obtain

$$\frac{dy}{dx}\frac{d}{dx}(\delta y) = \frac{d}{dx}\left(\frac{dy}{dx}\delta y\right) - \frac{d^2y}{dx^2}\delta y \tag{2.26}$$

Thus,

$$\delta E = \int_0^L \left[T\frac{d}{dx}\left(\frac{dy}{dx}\delta y\right) - T\frac{d^2y}{dx^2}\delta y - w\,\delta y \right] dx \tag{2.27}$$

The first term can be integrated to give

$$\delta E = T\frac{dy}{dx}\delta y\bigg|_0^L - \int_0^L \left[T\frac{d^2y}{dx^2}\delta y + w\,\delta y \right] dx \tag{2.28}$$

It is important to note two limitations that we must now place on the variation δy. First, in order to perform the integration indicated in the previous step, the integrand,

$$T\frac{d}{dx}\left(\frac{dy}{dx}\delta y\right)$$

must be integrable. Thus, δy must be a continuous function: otherwise its derivative would not exist at pints of discontinuity. Second, we must require that δy at the endpoints be zero; that is,

$$\delta y(0) = \delta y(L) = 0.0 \tag{2.29}$$

Otherwise the variation would create a function that did not satisfy the boundary conditions. Substitution of these limits of integration makes the first term in Eq. 2.28 equal to zero, and we are left with

$$\delta E = -\int_0^L \left[T\frac{d^2y}{dx^2} + w \right] \delta y\, dx \tag{2.30}$$

Note that if Eq. 2.19 is satisfied, then δE is equal to zero for any δy. Therefore, Eq. 2.19 is a sufficient condition to ensure that $y(x)$ gives a stationary value to E. On the other hand, if δE as given by Eq. 2.30 is zero for any δy (which satisfies the boundary conditions), then the term in braces must be identically zero. Hence, if we find a function $y(x)$ that gives a stationary value to E, then we have found the $y(x)$ that satisfies Eq. 2.19. An equation such as Eq. 2.19, when obtained in this manner, is referred to as the *Euler equation* of the corresponding functional.

A Second Approach. Although the above procedure for obtaining the equation for $y(x)$ that produces a stationary value of E is direct and easy to use, it is instructive, and often useful, to use the following approach.

Consider the potential energy written in terms of $y(x)$ (the function that gives it a stationary value) and its variation $\epsilon\eta$,[1]

$$E(y+\epsilon\eta) = \int_0^L \left[\frac{1}{2} T \left[\frac{d}{dx}(y+\epsilon\eta) \right]^2 - w(y+\epsilon\eta) \right] dx \tag{2.31}$$

Because E has a stationary value at $\epsilon = 0$, we know

$$\left. \frac{\partial E}{\partial \epsilon} \right|_{\epsilon=0} = 0 \tag{2.32}$$

for any $\eta(x)$. Expansion of Eq. 2.31 gives the following quadratic in ϵ.

$$\begin{aligned} E(y+\epsilon\eta) = & \left[\int_0^L \left[\frac{1}{2} T \left(\frac{dy}{dx} \right)^2 - wy \right] dx \right] \\ & + \left[\int_0^L \left[T \frac{dy}{dx} \frac{d\eta}{dx} - w\eta \right] dx \right] \epsilon \\ & + \left[\int_0^L \left[\frac{1}{2} T \left(\frac{d\eta}{dx} \right)^2 \right] dx \right] \epsilon^2 \end{aligned} \tag{2.33}$$

The derivative of E with respect to ϵ is

$$\begin{aligned} \frac{\partial E}{\partial \epsilon} = & \left[\int_0^L \left[T \frac{dy}{dx} \frac{d\eta}{dx} - w\eta \right] dx \right] \\ & + \left[\int_0^L \left[\frac{1}{2} T \left(\frac{d\eta}{dx} \right)^2 \right] dx \right] \epsilon \end{aligned} \tag{2.34}$$

If this is to be zero at $\epsilon = 0$, then

$$\int_0^L \left[T \frac{dy}{dx} \frac{d\eta}{dx} - w\eta \right] dx = 0 \tag{2.35}$$

You should now note the similarity between Eq. 2.35 and Eq. 2.24. These two equations impose identical requirements since the integrals must be zero for either an arbitrary δy or an arbitrary η. We therefore arrive at the same governing equation for $y(x)$ as we had before.

This second approach, however, gives us one bit of information that the first approach does not. Consider the expression for $E(y + \epsilon\eta)$ as given by Eq. 2.33. The second integral on the right-hand side we now know equals zero at the stationary value of E. Hence, $E(y + \epsilon\eta)$ is the sum of the E in the equilibrium position (the first integral) and an integral that is always positive. The stationary value of E is, therefore, a minimum value for the potential energy of the system.

[1]Remember, $\eta(0) = \eta(L) = 0$, so that the variation of y produces a new function that still satisfies the specified boundary conditions.

2.3 CORRESPONDING FUNCTIONALS

For the tight wire problem there existed a potential energy that produced a minimum value for the exact solution. Thus, we knew that for the differential equation

$$T\frac{d^2y}{dx^2} + w(x) = 0.0$$

there existed the corresponding functional

$$E(y) = \int_0^L \left[\frac{1}{2}T\left(\frac{dy}{dx}\right)^2 - wy\right]dx$$

In the previous section we demonstrated that for a given functional, the corresponding differential equation (the Euler equation) can easily be obtained. We now want to consider the following three questions:

1. Given a differential equation, is there always such a corresponding functional?
2. If there is a corresponding functional, can it be obtained from the differential equation?
3. If there is not a corresponding functional, can the finite element method be used to obtain approximate solutions to the governing equation?

Unfortunately, the answer to the first question is no; fortunately, however, the answer to the second and third questions is yes. To arrive at these answers we use the weak formulation as described in Chapter 1. Let us again go through the steps necessary to place the tight wire problem in its weak form, but now use the notation associated with the calculus of variations; that is, we will interpret the arbitrary weighting function as an arbitrary variation in y.

Given the function $y(x)$ that satisfies the above differential equation, $0 \leq x \leq L$, we know

$$\int_0^L \delta y\left(T\frac{d^2y}{dx^2} + w(x)\right)dx = 0 \tag{2.36}$$

must be satisfied for any variation of the solution δy. Thus,

$$\int_0^L \delta y\left(T\frac{d^2y}{dx^2} + w\right)dx = \int_0^L \left[\frac{d}{dx}\left(\delta y T\frac{dy}{dx}\right) - \frac{d\delta y}{dx}T\frac{dy}{dx} + w\,\delta y\right]dx \tag{2.37}$$

$$= \left[\delta y T\frac{dy}{dx}\right]_0^L - \int_0^L \left[\frac{d\delta y}{dx}T\frac{dy}{dx} - w\,\delta y\right]dx \tag{2.38}$$

$$= -\int_0^L \left[\frac{d\delta y}{dx}T\frac{dy}{dx} - w\,\delta y\right]dx \tag{2.39}$$

$$= -\int_0^L \left[\delta \frac{dy}{dx} T \frac{dy}{dx} - w \, \delta y \right] dx \tag{2.40}$$

$$= -\int_0^L \left[\delta \left(\frac{1}{2} T \left(\frac{dy}{dx} \right)^2 \right) - \delta(wy) \right] dx \tag{2.41}$$

$$= -\int_0^L \delta \left[\frac{1}{2} T \left(\frac{dy}{dx} \right)^2 - wy \right] dx \tag{2.42}$$

$$= -\delta \int_0^L \left[\frac{1}{2} T \left(\frac{dy}{dx} \right)^2 - wy \right] dx \tag{2.43}$$

$$= -\delta E = 0 \tag{2.44}$$

We have arrived at the corresponding variational principle of our problem, starting with the original differential equation. Similar steps can be taken to determine the functional corresponding to other differential equations—provided, of course, that a functional does exist.

Where, then, in the above steps was the crucial point that led to there being a functional? It occurred between Eqs. 2.40 and 2.41, where we rewrote each term in the integrand as a total variation of a single function. This allowed us to write the total integrand as a variation of a single function, and finally the entire expression as a variation of an integral.

There are, however, differential equations for which corresponding functionals such as that above do not exist. Such would be the case if, in our original linear differential equation, the first derivative of y had been present. In that case, the integrand in Eq. 2.40 would have included the term

$$\left(\frac{dy}{dx} \right) \delta y \quad \text{or simply} \quad y' \, \delta y$$

The fact that there is no function, say $J(y, y')$, whose total variation equals the above is easily shown. The total variation of J would be

$$\delta J = \frac{\partial J}{\partial y} \delta y + \frac{\partial J}{\partial y'} \delta y' \tag{2.45}$$

But this would have to equal

$$\delta J = y' \, \delta y \tag{2.46}$$

Comparison of the two forms for δJ shows that the partial of J with respect to y' must be zero; hence, J can not be a function of y'. This, however, contradicts Eq. 2.46; thus, J does not exist.

In comparison, consider that if our original equation had contained the function itself, e.g.,

$$\frac{d^2y}{dx^2} + y = 0$$

then $y\,\delta y$ would have appeared in the integrand. However, in this case, we could have written

$$y\,\delta y = \delta\left(\tfrac{1}{2}y^2\right)$$

to have created the necessary total variation.

We have thus demonstrated that when there is a corresponding functional, it can be derived from the differential equation; however, there may not always be a corresponding functional. Thus, we turn to our final question: If there is not a corresponding functional, can we still obtain an approximate solution by the finite element method? The answer, as stated above, is yes. We do not need a functional in order to use our finite element approximations. In fact, when there is such a functional, the first thing we do is take its variation to arrive at an equation similar to Eq. 2.40. It is this form into which we substitute our finite element approximations. Thus, the steps leading to Eqs. 2.41–2.44 are unnecessary for the development of our finite element equations. In the remainder of this book we will follow steps similar to those leading to Eq. 2.40 to formulate the governing equation used for the finite element approximation. It is this formulation that we will refer to as the weak form of the problem.

One final comment (question): Why, if we do not need a functional in order to develop our finite element equations, did we say it was unfortunate that the answer to our first question was no? There are two reasons. The first reason is that even if the functional is not directly used in the development of our equations, the fact that it exists gives us a better understanding of how our method converges to the solution (see comments at end of Chapter 1). The second reason is that when a functional does exist, the resulting finite element equations are described in terms of symmetric algebraic equations, whereas when a functional does not exist, the resulting equations are nonsymmetric.

EXERCISES

Study Problems

S1. Given

$$y = \sin(x) \quad \text{and} \quad \delta y = -0.5 + 0.2x^2$$

(a) Compare

$$\frac{d}{dx}(y + \delta y) - \frac{d}{dx}(y) \quad \text{with} \quad \frac{d}{dx}(\delta y)$$

(b) Compare

$$\int_0^3 (y + \delta y)dx - \int_0^3 (y)dx \quad \text{with} \quad \int_0^3 (\delta y)dx$$

S2. Given

$$F = 2y^2$$

let

$$y(x) = Y(x) = 6\sin(x)$$

Give a variation to $Y(x)$ equal to

$$\delta Y(x) = -\epsilon x^2$$

Determine by expanding all terms

$$\delta F = F(Y + \delta Y) - F(Y)$$

Write δF after all terms containing ϵ^2 and higher have been neglected. Now calculate

$$\frac{dF}{dy} \quad \text{at} \quad y = Y$$

Compare

$$\delta F = \frac{dF}{dy}\delta Y$$

with your previous equation for δF.

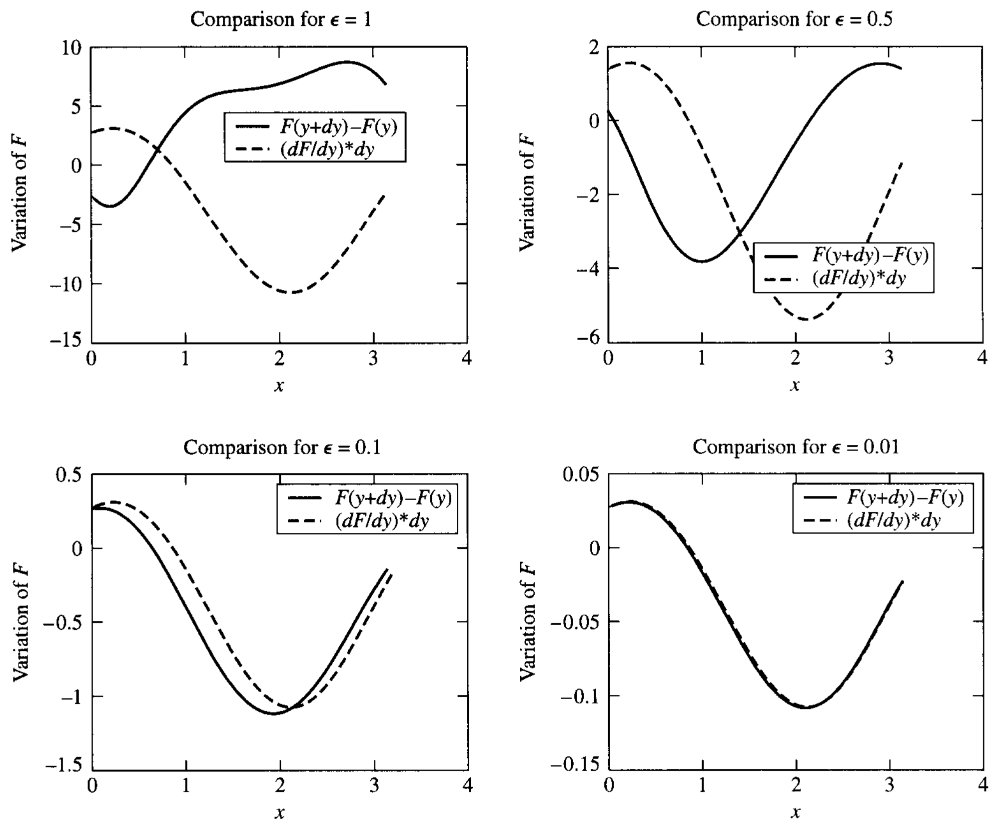
Numerical Experiments and Code Development

N1. Given

$$y = x$$
$$F(y) = y\cos(y)$$
$$\delta y = \epsilon\eta = \epsilon\left[5.0 - (x - 1.5)^2\right]$$

write a program to plot $F(y + \delta Y) - F(y)$ and $(dF/dy)\delta y$ as functions of x in the range $0 \leq x \leq \pi$ for $\epsilon = 1.0, 0.5, 0.1$, and 0.01. Discuss your solutions in terms of the approximations given in the text.

Answer:



$x \leq \pi$ for
he text.