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# Imposition of boundary conditions by modifying the weighting coefficient matrices in the differential quadrature method

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SUMMARY

One of the important issues in the implementation of the differential quadrature method is the imposition of the given boundary conditions. There may be multiple boundary conditions involving higher-order derivatives at the boundary points. The boundary conditions can be imposed by modifying the weighting coefficient matrices directly. However, the existing method is not robust and is known to have many limitations. In this paper, a systematic procedure is proposed to construct the modified weighting coefficient matrices to overcome these limitations. The given boundary conditions are imposed exactly. Furthermore, it is found that the numerical results depend only on those sampling grid points where the differential quadrature analogous equations of the governing differential equations are established. The other sampling grid points with no associated boundary conditions are not essential. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: higher-order differential equations; collocation method; modified weighting coefficient matrix; multiple boundary conditions; differential quadrature method

## 1. INTRODUCTION

The differential quadrature method (DQM) has been successfully used to tackle various initial and/or boundary value problems of physical and engineering science efficiently and accurately [1–4]. However, the imposition of the given initial/boundary conditions can be difficult when more than one boundary conditions are specified at a boundary point [1]. This situation is very commonly found in structural mechanics problems [5–8]. Bert and Malik [1] mentioned that this intriguing issue is not a straightforward matter and needs careful consideration.

### 1.1. The $\delta$-technique

Bert *et al.* [5] and Jang *et al.* [6] proposed a $\delta$-technique to impose the two given boundary conditions at each boundary point for structural mechanics problems. The $\delta$-technique consists

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of placing a series of two grid points separated from each other by a small distance $\delta$ near the boundary edge. One of the boundary conditions is applied at the grid points located on the boundary edge while the other is applied to the adjacent auxiliary $\delta$-grid points. It can be seen that one boundary condition is exactly imposed while the other is approximately imposed only.

### *1.2. Modified weighting coefficient matrices*

Wang and Bert [9], Wang *et al.* [10], and Malik and Bert [11] proposed several innovative methods to incorporate the boundary conditions by modifying the weighting coefficient matrices for structural mechanics problems. Civan [12] also incorporated the boundary conditions in the differential quadrature rule by replacing the derived differential quadrature rules by the conditions given at the boundary points. Shu and Xue [13] applied the same technique to impose the Neumann boundary conditions for incompressible Navier–Stokes equations. Malik and Civan [2] also considered this technique in the context of convection–diffusion–reaction problems. They all found that very accurate results could be obtained by the proposed techniques. However, Shu and Du [14, 15] also reported that these techniques had some major limitations and cannot be used to tackle general boundary conditions (for example, clamped and free support conditions). In fact, all these techniques may not produce reliable results since the interpolated numerical solutions may not satisfy the given boundary conditions exactly. In this paper, another procedure to modify the weighting coefficient matrices is proposed. The interpolated numerical solutions would satisfy the boundary conditions exactly.

### *1.3. Modified trial functions*

Alternatively, the boundary conditions involving higher-order derivatives can also be imposed exactly by modifying the trial functions to incorporate the degrees of freedom of the specified higher-order derivatives at the boundary [16–19], by using the differential quadrature element method [18–23], or by using the quadrature element method [24–26]. Basically, only Dirichlet-type and Neumann-type boundary conditions can be handled. The mixed-type boundary conditions cannot be tackled directly in general.

In this paper, the present modified weighting coefficient matrices are found to be equivalent to the weighting coefficient matrices given by Chen *et al.* in Reference [16] for Dirichlet-type and Neumann-type boundary conditions. However, the present procedure is computationally more efficient as no new trial functions satisfying the given boundary conditions have to be derived. Besides, the present procedure is more general and can tackle mixed-type nonhomogenous boundary conditions directly.

### *1.4. Differential quadrature analogous equations of the boundary conditions*

Another way to impose the boundary conditions is to apply the multiple boundary conditions at the same boundary points as given and to establish the differential quadrature analogous equations of the boundary conditions at the boundary points. It is different from the $\delta$-technique since the boundary conditions are not applied to the auxiliary $\delta$-grid points next to the boundary points. To eliminate the extra equations, the differential quadrature analogous equations of the governing differential equations at some selected sampling grid points are dropped. These selected points are called *auxiliary sampling grid points*. This approach has been used extensively by many researchers [7, 14, 15, 27–30]. It is found that this approach is a special case of the present algorithms when the same sampling grid points are used to establish the differential quadrature analogous equations of the governing differential equations. In Reference [14], this approach is viewed as substituting all the boundary conditions into the governing equations. In the present approach, more general non-homogenous boundary conditions are considered.

### 1.5. Essential and auxiliary sampling grid points

From the present formulation, it is found that the numerical results are in fact independent of the auxiliary sampling grid points. At these auxiliary sampling grid points, no differential quadrature analogous equations of the governing differential equations are established. In other words, the numerical results only depend on the *essential sampling grid points* where the differential quadrature analogous equations of the governing differential equations are established.

Shu and Chen [31] studied the solution accuracy when different sampling grid points were discarded. They concluded that the interior points just adjacent to the boundary should be discarded. In fact, they were studying the accuracy of the numerical solutions given by the remaining sampling grid points since all the sampling grid points (including the discarded sampling grid points) were Chebyshev–Gauss–Lobatto points. In this paper, it is advocated that only the remaining essential sampling grid points have to be the Chebyshev–Gauss–Lobatto-like points.

It is also noted that sometimes, after the boundary points and the auxiliary sampling grid points are discarded, the numerical solutions obtained by using the remaining sampling grid points to establish the differential quadrature analogous equations of the governing differential equations are not very accurate. It was suggested that the remaining sampling grid points should be stretched outward to give a better coverage [15, 31, 32]. Indeed, better numerical results were obtained. On the other hand, it was also reported that the numerical results could be sensitive to the distribution of the sampling grid points [33]. In the present formulation, this procedure is not necessary as the remaining essential sampling grid points are specified directly initially.

The manuscript is arranged as follows. The differential quadrature method is briefly reviewed in Section 2. In Section 3, the existing methods to impose the boundary conditions by modifying the weighting coefficient matrices are briefly discussed. In Section 4, the proposed method to impose the boundary conditions in the weighting coefficient matrices is presented. It is shown that the modified weighting coefficient matrices can be computed easily even for non-homogenous mixed-type boundary conditions involving higher-order derivatives. The roles of the essential and auxiliary sampling grid points are discussed in Section 5. The modified weighting coefficient matrices for second- and fourth-order equations are considered in Sections 6 and 7. Numerical examples and further discussions are given in Section 8. Conclusions are then given in Section 9.

## 2. DIFFERENTIAL QUADRATURE METHOD

In the differential quadrature method, the values of the derivatives at each sampling grid point are expressed as weighted linear sums of the function values at all sampling grid points within the domain under consideration. In other words, the $r$th derivative of the function $\Psi(x)$ at a sampling grid point $x = x_i$ is related to the function value $\Psi_k = \Psi(x_k)$

 

at $x = x_k$ by

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\Psi(x)\bigg|_{x=x_i} = \sum_{k=1}^{n} A_{ik}^{(r)}\Psi(x_k) \quad \text{for } i = 1, 2, \ldots, n \tag{1}$$

where $n$ is the total number of sampling grid points under consideration. $A_{ik}^{(r)}$ can be collectively written in a matrix form as

$$[\mathbf{A}^{(r)}] = \begin{bmatrix} A_{11}^{(r)} & A_{12}^{(r)} & \cdots & A_{1n}^{(r)} \\ A_{21}^{(r)} & A_{22}^{(r)} & \cdots & A_{2n}^{(r)} \\ \vdots & \vdots & & \vdots \\ A_{n1}^{(r)} & A_{n2}^{(r)} & \cdots & A_{nn}^{(r)} \end{bmatrix} \tag{2}$$

and $[\mathbf{A}^{(r)}]$ is called the weighting coefficient matrix. The evaluation of the weighting coefficient matrix has been discussed extensively [1].

Consider a linear $m$th order ordinary differential equation in the form

$$\alpha_0 \frac{\mathrm{d}^m y}{\mathrm{d}x^m} + \alpha_1 \frac{\mathrm{d}^{m-1} y}{\mathrm{d}x^{m-1}} + \cdots + \alpha_m y = f(x) \quad \text{for } 0 < x < L \text{ and } \alpha_0 \neq 0 \tag{3}$$

Using the relation in Equation (1), the differential quadrature analogous equations of the governing differential equations at the $n$ sampling grid points $x_1, x_2, \ldots, x_n$ can be written as

$$(\alpha_0[\mathbf{A}^{(m)}] + \alpha_1[\mathbf{A}^{(m-1)}] + \cdots + \alpha_{m-1}[\mathbf{A}^{(1)}] + \alpha_m[\mathbf{A}^{(0)}])\{\mathbf{Y}\} = \{\mathbf{f}\} \tag{4}$$

where

$$[\mathbf{A}^{(0)}] = [\mathbf{I}], \quad \{\mathbf{Y}\} = \begin{Bmatrix} y_1 \\ \vdots \\ y_n \end{Bmatrix}, \quad \{\mathbf{f}\} = \begin{Bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{Bmatrix} \tag{5}$$

and $y_1, y_2, \ldots, y_n$ are the approximate values of $y(x)$ at $x_1, x_2, \ldots, x_n$, respectively.

Of course, Equation (4) cannot be solved until the boundary conditions are imposed properly. For an $m$th order equation, there should be $m$ boundary conditions. The solution procedure can be implemented in several ways:

(i) Select $n - m$ equations from Equation (4) and construct the $m$ differential quadrature analogous equations of the boundary conditions at the boundary points. The $n$ unknowns $y_1, y_2, \ldots, y_n$ are then solved from the combined $n$ equations. This method is very commonly used [7, 14, 15, 27–30].

(ii) Select $n - m$ equations from Equation (4) and construct the $m$ differential quadrature analogous equations of the boundary conditions at the boundary and adjacent points to solve for $y_1, y_2, \ldots, y_n$ [5, 6]. This is the $\delta$-technique and the boundary conditions are only satisfied approximately. This method is also very commonly used [33–39].

(iii) Construct the weighting coefficient matrices $[\mathbf{A}^{(r)}]$ from the trial functions that satisfy the given boundary conditions exactly [16–26]. This method is equivalent to the collocation method.

 

(iv) Modify the weighting coefficient matrices $[\mathbf{A}^{(r)}]$ to incorporate the given boundary conditions. The boundary conditions are satisfied approximately by the interpolated solutions [9–13].

(v) Modify the weighting coefficient matrices $[\mathbf{A}^{(r)}]$ to incorporate the given boundary conditions. The boundary conditions are satisfied exactly by the interpolated solutions (the present procedure).

In Methods (ii) and (iv), the interpolated solutions satisfy the given boundary conditions approximately only. It can be shown that Methods (i), (iii), and (v) are equivalent if the same sampling grid points are used to establish the differential quadrature analogous equations of the governing differential equations. However, it is difficult to construct the required trial functions that satisfy the mixed type boundary conditions for Method (iii). For Method (i), $n$ unknowns are solved simultaneously while in Method (v), only $n-m$ unknowns are solved. The significance of Method (v) is that, from the final forms of the modified weighting coefficient matrices, it is realized that the numerical results are in fact independent of the auxiliary sampling grid points. The selection of the essential and auxiliary sampling grid points should be viewed under a different perspective.

## 3. EXISTING APPROACHES TO MODIFY THE WEIGHTING COEFFICIENT MATRICES

In Wang and Bert [9], and Malik and Bert [11], the boundary conditions are incorporated by modifying the weighting coefficient matrices. For a simply supported beam, the boundary conditions at the two ends can be expressed as $y(x_1)=0$, $y''(x_1)=0$, $y(x_n)=0$ and $y''(x_n)=0$. In Reference [9], to impose the boundary conditions in the weighting coefficient matrices, all the elements in the columns corresponding to $x_1$ and $x_n$ in the weighting coefficient matrix $[\mathbf{A}^{(1)}]$ are set to zero, i.e. $[\mathbf{A}^{(1)}]$ is modified to $[\tilde{\mathbf{A}}^{(1)}]$ as

$$[\tilde{\mathbf{A}}^{(1)}]=\begin{bmatrix} 0 & A_{12}^{(1)} & \cdots & A_{1,n-1}^{(1)} & 0 \\ 0 & A_{22}^{(1)} & \cdots & A_{2,n-1}^{(1)} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & A_{n2}^{(1)} & \cdots & A_{n,n-1}^{(1)} & 0 \end{bmatrix} \tag{6}$$

The weighting coefficient matrices for the second derivative $[\tilde{\mathbf{A}}^{(2)}]$ and the fourth derivative $[\tilde{\mathbf{A}}^{(4)}]$ are then obtained from

$$[\tilde{\mathbf{A}}^{(2)}]=[\mathbf{A}^{(1)}][\tilde{\mathbf{A}}^{(1)}] \tag{7a}$$

and

$$[\tilde{\mathbf{A}}^{(4)}]=[\tilde{\mathbf{A}}^{(2)}][\tilde{\mathbf{A}}^{(2)}] \tag{7b}$$

This procedure is applicable to the simply supported boundary condition only. A more general discussion on the imposition of other types of boundary conditions was given in

 

Reference [11]. Essentially, starting from $[\bar{\mathbf{A}}^{(1)}]=[\mathbf{A}^{(1)}]$, a new weighting coefficient matrix $[\tilde{\mathbf{A}}^{(r)}]$ for the $r$th derivative with boundary conditions built-in is to be obtained from a derived weighting coefficient matrix $[\bar{\mathbf{A}}^{(r)}]$. If a boundary condition is expressed in the form

$$\left.\frac{\mathrm{d}^r y}{\mathrm{d}x^r}\right|_{x=x_j}=\sum_{p=0}^{r-1}\gamma_p\left.\frac{\mathrm{d}^p y}{\mathrm{d}x^p}\right|_{x=x_j} \qquad \text{where } m-1\geqslant r\geqslant 1 \tag{8}$$

then all the elements in $[\tilde{\mathbf{A}}^{(r)}]$ and $[\bar{\mathbf{A}}^{(r)}]$ are basically the same (i.e. $\tilde{A}^{(r)}_{jk}=\bar{A}^{(r)}_{jk}$) except all the elements in the $j$th row. In particular, the $k$th element in the $j$th row is given by

$$\tilde{A}^{(r)}_{jk}=\sum_{p=0}^{r-1}\gamma_p\tilde{A}^{(p)}_{jk} \tag{9}$$

where $\tilde{A}^{(p)}_{jk}$ are elements in $[\tilde{\mathbf{A}}^{(p)}]$ from the previously modified weighting coefficient matrices with the appropriate boundary conditions imposed.

Once the modified weighting coefficient matrix $[\tilde{\mathbf{A}}^{(r)}]$ is obtained, the weighting coefficient matrix for the $(r+1)$th order derivative $[\bar{\mathbf{A}}^{(r+1)}]$ can be derived from

$$[\bar{\mathbf{A}}^{(r+1)}]=[\mathbf{A}^{(1)}][\tilde{\mathbf{A}}^{(r)}] \tag{10}$$

At this stage, additional boundary conditions involving the $(r+1)$th order derivatives in the form of Equation (8) can be imposed by using Equation (9) again. The process continues until all the boundary conditions are imposed and the required modified weighting coefficient matrices $[\tilde{\mathbf{A}}^{(1)}],\ldots,[\tilde{\mathbf{A}}^{(m)}]$ are obtained. In other words, Equations (9) and (10) are used alternatively as follows:

$$[\mathbf{A}^{(1)}]\rightarrow[\tilde{\mathbf{A}}^{(1)}]\rightarrow[\bar{\mathbf{A}}^{(2)}]\rightarrow[\tilde{\mathbf{A}}^{(2)}]\rightarrow\cdots[\bar{\mathbf{A}}^{(m)}]\rightarrow[\tilde{\mathbf{A}}^{(m)}] \tag{11}$$

The differential quadrature analogous equations of the governing differential equations can then be established as Equation (4) with $[\mathbf{A}^{(r)}]$ replaced by $[\tilde{\mathbf{A}}^{(r)}]$. All the boundary conditions except the Dirichlet-type boundary conditions (i.e. $y(x_i)=\beta_i$) are imposed. Hence, before Equation (4) is solved, the differential quadrature analogous equations corresponding to the sampling grid points with Dirichlet-type boundary conditions are dropped and the unknowns are replaced by the given values in the remaining equations.

Civan [12] has extended the procedure to incorporate non-homogenous boundary conditions in Equation (8). Some implementation details can be found in Reference [12] and in the numerical example in Section 8.5.

For a simply supported beam, it can be shown that the resultant matrices $[\tilde{\mathbf{A}}^{(4)}]$ given by Equation (11) and $[\tilde{\mathbf{A}}^{(4)}]$ given by Equation (7b) after removing the columns and rows corresponding to the supports are in fact the same. It was also found that the numerical results so obtained were very accurate [9, 11]. However, it is also known that the numerical results may not be reliable, as reported by Shu and Du [14]. For free vibration analysis, there may be additional zero and/or phantom eigenvalues. As a result, the procedure to impose the boundary conditions into the weighting coefficient matrices has to be reviewed.

## 4. PRESENT APPROACH TO MODIFY THE WEIGHTING COEFFICIENT MATRICES

Let the $m$ boundary conditions for the differential equation in Equation (3) be given in the following non-homogenous mixed form as

$$\gamma_{i1} \left.\frac{\mathrm{d}^{m-1}y}{\mathrm{d}x^{m-1}}\right|_{x=\bar{x}_i} + \gamma_{i2} \left.\frac{\mathrm{d}^{m-2}y}{\mathrm{d}x^{m-2}}\right|_{x=\bar{x}_i} + \cdots + \gamma_{im} y(\bar{x}_i) = \beta_i \quad \text{for } i = 1, 2, \ldots, m \tag{12}$$

where $\gamma_{ij}$ and $\beta_i$ are the constant coefficients (and some of them may be zero), $\bar{x}_1, \ldots, \bar{x}_m$ are the co-ordinates of the boundary points ($\bar{x}_1, \ldots, \bar{x}_m$ may not be all distinct). For example, for initial value problems, $\bar{x}_1 = \cdots = \bar{x}_m = 0$ corresponds to the initial starting point [40–43]. For boundary value problems, $\bar{x}_i$ would be 0 or 1. In a more general situation, $\bar{x}_i$ can be arbitrary within the interval.

If the $n$ differential quadrature analogous equations of the governing differential equations in Equation (4) are to be used, additional auxiliary sampling grid points are required. Let the $m$ additional auxiliary sampling grid points be $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$. The extended differential quadrature rules are still given by Equation (1) with $n$ replaced by $n+m$, i.e.

$$\left\{\begin{array}{c} y_1^{(r)} \\ \vdots \\ y_n^{(r)} \\ \hline y_{n+1}^{(r)} \\ \vdots \\ y_{n+m}^{(r)} \end{array}\right\} = \left[\begin{array}{ccc|ccc} A_{11}^{(r)} & \cdots & A_{1n}^{(r)} & A_{1,n+1}^{(r)} & \cdots & A_{1,n+m}^{(r)} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1}^{(r)} & \cdots & A_{nn}^{(r)} & A_{n,n+1}^{(r)} & \cdots & A_{n,n+m}^{(r)} \\ \hline A_{n+1,1}^{(r)} & \cdots & A_{n+1,n}^{(r)} & A_{n+1,n+1}^{(r)} & \cdots & A_{n+1,n+m}^{(r)} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n+m,1}^{(r)} & \cdots & A_{n+m,n}^{(r)} & A_{n+m,n+1}^{(r)} & \cdots & A_{n+1,n+m}^{(r)} \end{array}\right] \left\{\begin{array}{c} y_1 \\ \vdots \\ y_n \\ \hline y_{n+1} \\ \vdots \\ y_{n+m} \end{array}\right\} \tag{13}$$

or

$$\{\mathbf{Y}_1^{(r)}\} = [\mathbf{A}_1^{(r)} \quad \mathbf{A}_2^{(r)}] \left\{\begin{array}{c} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{array}\right\} \tag{14a}$$

and

$$\{\mathbf{Y}_2^{(r)}\} = [\mathbf{A}_3^{(r)} \ \ \mathbf{A}_4^{(r)}] \left\{\begin{array}{c} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{array}\right\} \tag{14b}$$

where

$$\{\mathbf{Y}_1^{(r)}\} = \left\{\begin{array}{c} y_1^{(r)} \\ \vdots \\ y_n^{(r)} \end{array}\right\}, \quad \{\mathbf{Y}_2^{(r)}\} = \left\{\begin{array}{c} y_{n+1}^{(r)} \\ \vdots \\ y_{n+m}^{(r)} \end{array}\right\}, \quad \{\mathbf{Y}_1\} = \left\{\begin{array}{c} y_1 \\ \vdots \\ y_n \end{array}\right\} \quad \{\mathbf{Y}_2\} = \left\{\begin{array}{c} y_{n+1} \\ \vdots \\ y_{n+m} \end{array}\right\}, \quad y_j^{(r)} = \left.\frac{\mathrm{d}^r y}{\mathrm{d}x^r}\right|_{x=x_j} \tag{15a}$$

$$\{\mathbf{Y}_1^{(0)}\} = \{\mathbf{Y}_1\}, \quad [\mathbf{A}_1^{(r)}] = \begin{bmatrix} A_{11}^{(r)} & \cdots & A_{1n}^{(r)} \\ \vdots & & \vdots \\ A_{n1}^{(r)} & \cdots & A_{nn}^{(r)} \end{bmatrix}, \quad [\mathbf{A}_2^{(r)}] = \begin{bmatrix} A_{1,n+1}^{(r)} & \cdots & A_{1,n+m}^{(r)} \\ \vdots & & \vdots \\ A_{n,n+1}^{(r)} & \cdots & A_{n,n+m}^{(r)} \end{bmatrix} \tag{15b}$$

$$\{\mathbf{Y}_2^{(0)}\} = \{\mathbf{Y}_2\}, \quad [\mathbf{A}_3^{(r)}] = \begin{bmatrix} A_{n+1,1}^{(r)} & \cdots & A_{n+1,n}^{(r)} \\ \vdots & & \vdots \\ A_{n+m,1}^{(r)} & \cdots & A_{n+m,n}^{(r)} \end{bmatrix} \quad [\mathbf{A}_4^{(r)}] = \begin{bmatrix} A_{n+1,n+1}^{(r)} & \cdots & A_{n+1,n+m}^{(r)} \\ \vdots & & \vdots \\ A_{n+m,n+1}^{(r)} & \cdots & A_{n+m,n+m}^{(r)} \end{bmatrix} \tag{15c}$$

Note that in establishing Equation (4), only $[\mathbf{A}_1^{(r)}]$ and $[\mathbf{A}_2^{(r)}]$ are required. However, some of the values in $[\mathbf{A}_3^{(r)}]$ and $[\mathbf{A}_4^{(r)}]$ are also required to establish the differential quadrature analogous equations of the boundary conditions.

The differential quadrature analogous equations of the boundary condition in Equation (12) can be written as

$$[\gamma_{i1}\ \gamma_{i2}\ \cdots\ \gamma_{im}] \left[\begin{array}{ccc|ccc} A_{j1}^{(m-1)} & \cdots & A_{jn}^{(m-1)} & A_{j,n+1}^{(m-1)} & \cdots & A_{j,n+m}^{(m-1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{j1}^{(0)} & \cdots & A_{jn}^{(0)} & A_{j,n+1}^{(0)} & \cdots & A_{j,n+m}^{(0)} \end{array}\right] \left\{\begin{array}{c} y_1 \\ \vdots \\ y_n \\ \hline y_{n+1} \\ \vdots \\ y_{n+m} \end{array}\right\} = \beta_i \tag{16}$$

where the boundary point $\bar{x}_i$ corresponds to one of the $x_j$ in $x_1, \ldots, x_{n+m}$ so that $A_{ij}^{(r)}$ can be obtained from Equation (13). It is possible that $\bar{x}_i$ is in fact included in $x_1, \ldots, x_n$ so that the boundary point is also one of the collocation points (essential sampling grid points).

The first two matrices in Equation (16) can be combined as

$$[\Gamma_{i1}\ \cdots\ \Gamma_{i,n}\ |\ \Gamma_{i,n+1}\ \cdots\ \Gamma_{i,n+m}] \left\{\begin{array}{c} y_1 \\ \vdots \\ y_n \\ \hline y_{n+1} \\ \vdots \\ y_{n+m} \end{array}\right\} = \beta_i \tag{17}$$

 

As a result, the $m$ boundary conditions can be collectively written as

$$\left[\begin{array}{ccc|ccc} \Gamma_{11} & \cdots & \Gamma_{1,n} & \Gamma_{1,n+1} & \cdots & \Gamma_{1,n+m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Gamma_{m1} & \cdots & \Gamma_{m,n} & \Gamma_{m,n+1} & \cdots & \Gamma_{m,n+m} \end{array}\right] \left\{\begin{array}{c} y_1 \\ \vdots \\ y_n \\ \hline y_{n+1} \\ \vdots \\ y_{n+m} \end{array}\right\} = \left\{\begin{array}{c} \beta_1 \\ \vdots \\ \beta_m \end{array}\right\}$$

or

$$[\boldsymbol{\Gamma}_1]\{\mathbf{Y}_1\} + [\boldsymbol{\Gamma}_2]\{\mathbf{Y}_2\} = \{\boldsymbol{\beta}\} \tag{18}$$

Hence, from Equation (18), if $[\boldsymbol{\Gamma}_2]$ is non-singular, $\{\mathbf{Y}_2\}$ can be expressed as

$$\{\mathbf{Y}_2\} = -[\boldsymbol{\Gamma}_2]^{-1}[\boldsymbol{\Gamma}_1]\{\mathbf{Y}_1\} + [\boldsymbol{\Gamma}_2]^{-1}\{\boldsymbol{\beta}\} \tag{19}$$

Equation (14a) then becomes

$$\begin{aligned} \{\mathbf{Y}_1^{(r)}\} &= ([\mathbf{A}_1^{(r)}] - [\mathbf{A}_2^{(r)}][\boldsymbol{\Gamma}_2]^{-1}[\boldsymbol{\Gamma}_1])\{\mathbf{Y}_1\} + [\mathbf{A}_2^{(r)}][\boldsymbol{\Gamma}_2]^{-1}\{\boldsymbol{\beta}\} \\ &= [\tilde{\mathbf{A}}^{(r)}]\{\mathbf{Y}_1\} + [\tilde{\mathbf{B}}^{(r)}]\{\boldsymbol{\beta}\} \end{aligned} \tag{20}$$

Equation (20) is the modified differential quadrature rule with the non-homogenous mixed-type boundary conditions in Equation (12) imposed. The interpolated solutions would satisfy the boundary conditions exactly. $[\tilde{\mathbf{A}}^{(r)}]$ and $[\tilde{\mathbf{B}}^{(r)}]$ are the modified weighting coefficient matrix and the coefficient matrix for the non-homogenous terms, respectively.

Equation (4) then becomes

$$\left(\sum_{r=0}^{m} \alpha_r[\tilde{\mathbf{A}}^{(m-r)}]\right)\{\mathbf{Y}_1\} = \{\mathbf{f}\} - \left(\sum_{r=0}^{m} \alpha_r[\tilde{\mathbf{B}}^{(m-r)}]\right)\{\boldsymbol{\beta}\} \tag{21}$$

$\{\mathbf{Y}_1\}$ can be solved from Equation (21).

## 5. SAMPLING GRID POINTS

In choosing the auxiliary sampling grid points, $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$, it is important that all the sampling grid points $x_1, x_2, \ldots, x_{n+m}$ must be distinct. Otherwise the weighting coefficients $A_{ij}^{(r)}$ in Equation (13) cannot be evaluated. Besides, the boundary point $\bar{x}_i$ should be included in $x_1, \ldots, x_{n+m}$. Otherwise, the weighting coefficients $A_{ij}^{(r)}$ used in Equation (16) may have to be determined separately. As a result, if the boundary point $\bar{x}_i$ is not in $x_1, x_2, \ldots, x_n$, then $\bar{x}_i$ should be included in the auxiliary sampling grid points $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$. This arrangement would facilitate the computation of the modified weighting coefficient matrices in Equation (20).

It can be verified that the modified weighting coefficient matrices $[\tilde{\mathbf{A}}^{(r)}]$ and $[\tilde{\mathbf{B}}^{(r)}]$ in Equation (20) are in fact independent of the non-boundary auxiliary sampling grid points.

 

In other words, the actual values of the auxiliary sampling grid points are not important as long as there is no numerical stability problem in evaluating the weighting coefficients $A_{ij}^{(r)}$ in Equation (13). As a result, the auxiliary sampling grid points need not be too close to the other sampling grid points at the boundary. Hence, the $\delta$-technique is not really necessary.

Tomasiello [44] had also observed this independence in his numerical calculations. He had chosen 0, $b_2$, $b_3$, $1-b_3$, $1-b_2$ and 1 as the sampling grid points. However, he found that the third co-ordinate $b_3$ played an important role in obtaining good results while varying the co-ordinate of the second node $b_2$ did not have much influence on the numerical results. It can be checked that $b_2$ is in fact a non-boundary auxiliary node.

This finding is not too surprising, as the differential quadrature method is in fact equivalent to the collocation method. In the collocation method, only the collocation points (equivalent to the present essential sampling grid points $x_1, x_2, \ldots, x_n$) are specified while the trial functions satisfy the given boundary conditions initially. The weighting coefficients are then obtained by differentiating the trial functions and then substituting the co-ordinates of the sampling grid points into the resultant expressions. There are no auxiliary sampling grid points in the formulation. Alternatively, the weighting coefficient matrices can also be obtained by matrix manipulation [16]. Again, it can be seen that there are no auxiliary sampling grid points. In both cases, the trial functions are polynomial of degree $n+m-1$.

On the other hand, in the present formulation, the trial functions are expressed as Lagrange polynomials with sampling grid points at $x_1, x_2, \ldots, x_{n+m}$. The polynomials are of degree $n+m-1$ as well. If the Lagrange polynomials also satisfy the boundary conditions, then the polynomials would be equivalent to the trial functions used in the collocation method. Hence, the two algorithms are in fact equivalent. The present formulation is simpler as the trial functions satisfying the given boundary conditions need not be determined first before the calculation of the weighting coefficients can be carried out. Besides, the determination of the trial functions could be quite difficult for mixed-type boundary conditions.

In the following, the formulations for the second- and fourth-order equations are considered.

## 6. SECOND-ORDER EQUATIONS

Consider a second-order equation in the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha_1 \frac{\mathrm{d}y}{\mathrm{d}x} + \alpha_2 y = f(x) \quad \text{for } 0 < x < 1 \tag{22}$$

with general mixed-type boundary conditions

$$\Gamma_{11}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=0} + \Gamma_{12}y(0) = \beta_1 \quad \text{at } x=0 \tag{23a}$$

and

$$\Gamma_{21}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=1} + \Gamma_{22}y(1) = \beta_2 \quad \text{at } x=1 \tag{23b}$$

where $\Gamma_{11}$, $\Gamma_{12}$, $\Gamma_{21}$, $\Gamma_{22}$, $\beta_1$, and $\beta_2$ are constant values. Since the boundary points $x=0$ and 1 have to be included in the sampling grid points, let the $n+2$ sampling grid points be arranged in an ascending order as $0 = x_1 < x_2 < \cdots x_{n+1} < x_{n+2} = 1$. The differential quadrature rules are given by

$$\begin{Bmatrix} y_1^{(r)} \\ \vdots \\ y_{n+2}^{(r)} \end{Bmatrix} = \begin{bmatrix} A_{11}^{(r)} & \cdots & A_{1,n+2}^{(r)} \\ \vdots & & \vdots \\ A_{n+2,1}^{(r)} & \cdots & A_{n+2,n+2}^{(r)} \end{bmatrix} \begin{Bmatrix} y_1 \\ \vdots \\ y_{n+2} \end{Bmatrix} \tag{24}$$

Similarly, the two boundary conditions can be written as

$$\begin{bmatrix} \Gamma_{11}A_{11}^{(1)} + \Gamma_{12} & \Gamma_{11}A_{12}^{(1)} & \cdots & \Gamma_{11}A_{1,n+1}^{(1)} & \Gamma_{11}A_{1,n+2}^{(1)} \\ \Gamma_{21}A_{n+2,1}^{(1)} & \Gamma_{21}A_{n+2,2}^{(1)} & \cdots & \Gamma_{21}A_{n+2,n+1}^{(1)} & \Gamma_{21}A_{n+2,n+2}^{(1)} + \Gamma_{22} \end{bmatrix} \begin{Bmatrix} y_1 \\ \vdots \\ y_{n+2} \end{Bmatrix} = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \tag{25}$$

Any two unknowns from $y_1, \ldots, y_{n+2}$ can be eliminated using Equation (25). In general, the sampling grid points $x_2, \ldots, x_{n+1}$ will be chosen to establish the differential quadrature analogous equations of the governing differential equations. The two end points $x_1$ and $x_{n+2}$ are then treated as auxiliary sampling grid points. In this case, $y_1$ and $y_{n+2}$ can be expressed in terms of $y_2, \ldots, y_{n+1}$ as

$$\begin{Bmatrix} y_1 \\ y_{n+2} \end{Bmatrix} = \begin{bmatrix} \Gamma_{11}A_{11}^{(1)} + \Gamma_{12} & \Gamma_{11}A_{1,n+2}^{(1)} \\ \Gamma_{21}A_{n+2,1}^{(1)} & \Gamma_{21}A_{n+2,n+2}^{(1)} + \Gamma_{22} \end{bmatrix}^{-1} \times \left( \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} - \begin{bmatrix} \Gamma_{11}A_{12}^{(1)} & \cdots & \Gamma_{11}A_{1,n+1}^{(1)} \\ \Gamma_{21}A_{n+2,2}^{(1)} & \cdots & \Gamma_{21}A_{n+2,n+1}^{(1)} \end{bmatrix} \begin{Bmatrix} y_2 \\ \vdots \\ y_{n+1} \end{Bmatrix} \right) \tag{26}$$

The differential quadrature rules at $x_2, \ldots, x_{n+1}$ in Equation (24) are then written as

$$\begin{Bmatrix} y_2^{(r)} \\ \vdots \\ y_{n+1}^{(r)} \end{Bmatrix} = \left[\tilde{\mathbf{A}}^{(r)}\right] \begin{Bmatrix} y_2 \\ \vdots \\ y_{n+1} \end{Bmatrix} + [\tilde{\mathbf{B}}^{(r)}] \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \tag{27}$$

where

$$[\tilde{\mathbf{A}}^{(r)}] = \begin{bmatrix} A_{22}^{(r)} & \cdots & A_{2,n+1}^{(r)} \\ \vdots & & \vdots \\ A_{n+1,2}^{(r)} & \cdots & A_{n+1,n+1}^{(r)} \end{bmatrix} - [\tilde{\mathbf{B}}^{(r)}] \begin{bmatrix} \Gamma_{11}A_{12}^{(1)} & \cdots & \Gamma_{11}A_{1,n+1}^{(1)} \\ \Gamma_{21}A_{n+2,2}^{(1)} & \cdots & \Gamma_{21}A_{n+2,n+1}^{(1)} \end{bmatrix} \tag{28a}$$

 

$$[\tilde{\mathbf{B}}^{(r)}] = \begin{bmatrix} A_{21}^{(r)} & A_{2,n+2}^{(r)} \\ \vdots & \vdots \\ A_{n+1,1}^{(r)} & A_{n+1,n+2}^{(r)} \end{bmatrix} \begin{bmatrix} \Gamma_{11}A_{11}^{(1)} + \Gamma_{12} & \Gamma_{11}A_{1,n+2}^{(1)} \\ \Gamma_{21}A_{n+2,1}^{(1)} & \Gamma_{21}A_{n+2,n+2}^{(1)} + \Gamma_{22} \end{bmatrix}^{-1} \tag{28b}$$

### *6.1. Weighting coefficient matrices from the collocation method*

Alternatively, the differential quadrature rules in Equation (27) can be obtained from the collocation method as well. The trial functions have to satisfy the boundary conditions in Equation (23). Let

$$y(x) = y_1 L_1^{n+1}(x) + y_2 J_2(x) + \cdots + y_{n+1} J_{n+1}(x) + y_{n+2} L_{n+2}^{n+1}(x) \tag{29}$$

where

$$J_k(x) = L_k^{n+1}(x) + a_k L_1^{n+1}(x) + b_k L_{n+2}^{n+1}(x) \quad \text{for } k = 2, 3, \ldots, n+1 \tag{30}$$

$a_k$ and $b_k$ are some undetermined coefficients, and $L_k^{n+1}(x)$ is the $(n+1)$th order Lagrange polynomial and is given by

$$L_k^{n+1}(x) = \prod_{\substack{j=1 \\ j\neq k}}^{n+2} \frac{x - x_j}{x_k - x_j} \quad \text{for } 1 \leqslant k \leqslant n+2 \tag{31}$$

It can be seen that $J_k(x_k) = 1$ and $J_k(x_j) = 0$ when $k \neq j$ for $2 \leqslant j \leqslant n+1$ and $2 \leqslant k \leqslant n+1$.

Furthermore, $a_k$ and $b_k$ can be determined such that

$$\Gamma_{11} J_k'(0) + \Gamma_{12} J_k(0) = 0 \quad \text{and} \quad \Gamma_{21} J_k'(1) + \Gamma_{22} J_k(1) = 0 \tag{32}$$

Since

$$\begin{aligned} J_k(0) &= a_k, \quad J_k'(0) = A_{1k}^{(1)} + a_k A_{11}^{(1)} + b_k A_{1,n+2}^{(1)} \\ J_k(1) &= b_k, \quad J_k'(1) = A_{n+2,k}^{(1)} + a_k A_{n+2,1}^{(1)} + b_k A_{n+2,n+2}^{(1)} \end{aligned} \tag{33}$$

$a_k$ and $b_k$ can be evaluated as

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = - \begin{bmatrix} \Gamma_{11}A_{11}^{(1)} + \Gamma_{12} & \Gamma_{11}A_{1,n+2}^{(1)} \\ \Gamma_{21}A_{n+2,1}^{(1)} & \Gamma_{21}A_{n+2,n+2}^{(1)} + \Gamma_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \Gamma_{11}A_{1k}^{(1)} \\ \Gamma_{21}A_{n+2,k}^{(1)} \end{Bmatrix} \tag{34}$$

$y_1$ and $y_{n+2}$ are determined such that the boundary conditions are satisfied, i.e.

$$\begin{aligned} \Gamma_{11} y'(0) + \Gamma_{12} y(0) &= \Gamma_{11}(y_1 A_{11}^{(1)} + y_{n+2} A_{1,n+2}^{(1)}) + \Gamma_{12} y_1 = \beta_1 \\ \Gamma_{21} y'(1) + \Gamma_{22} y(1) &= \Gamma_{21}(y_1 A_{n+2,1}^{(1)} + y_{n+2} A_{n+2,n+2}^{(1)}) + \Gamma_{22} y_{n+2} = \beta_2 \end{aligned} \tag{35}$$

or

$$\begin{Bmatrix} y_1 \\ y_{n+2} \end{Bmatrix} = \begin{bmatrix} \Gamma_{11}A_{11}^{(1)} + \Gamma_{12} & \Gamma_{11}A_{1,n+2}^{(1)} \\ \Gamma_{21}A_{n+2,1}^{(1)} & \Gamma_{21}A_{n+2,n+2}^{(1)} + \Gamma_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \tag{36}$$

 

The differential quadrature rules in Equation (27) can be obtained by differentiating Equation (29) with respect to $x$ and substituting the values of $x = x_2, \ldots, x_{n+1}$ one by one. It can be shown that the results are in fact equivalent to Equations (28a) and (28b). However, the present procedure in Equation (28) is easier to implement.

Of course, other $n$ sampling grid points from $x_1, \ldots, x_{n+2}$ may be chosen. For example, if $x_1, \ldots, x_n$ are chosen (i.e. eliminating $x_{n+1}$ and $x_{n+2}$), then the differential quadrature rules are then written as

$$\begin{Bmatrix} y_1^{(r)} \\ \vdots \\ y_n^{(r)} \end{Bmatrix} = [\tilde{\mathbf{A}}^{(r)}] \begin{Bmatrix} y_1 \\ \vdots \\ y_n \end{Bmatrix} + [\tilde{\mathbf{B}}^{(r)}] \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \tag{37}$$

where

$$[\tilde{\mathbf{A}}^{(r)}] = \begin{bmatrix} A_{11}^{(r)} & \cdots & A_{1,n}^{(r)} \\ \vdots & & \vdots \\ A_{n,1}^{(r)} & \cdots & A_{n,n}^{(r)} \end{bmatrix} - [\tilde{\mathbf{B}}^{(r)}] \begin{bmatrix} \Gamma_{11}A_{1,1}^{(1)} + \Gamma_{12} & \Gamma_{11}A_{1,2}^{(1)} & \cdots & \Gamma_{11}A_{1,n}^{(1)} \\ \Gamma_{21}A_{n+2,1}^{(1)} & \Gamma_{21}A_{n+2,2}^{(1)} & \cdots & \Gamma_{21}A_{n+2,n}^{(1)} \end{bmatrix} \tag{38a}$$

$$[\tilde{\mathbf{B}}^{(r)}] = \begin{bmatrix} A_{1,n+1}^{(r)} & A_{1,n+2}^{(r)} \\ \vdots & \vdots \\ A_{n,n+1}^{(r)} & A_{n,n+2}^{(r)} \end{bmatrix} \begin{bmatrix} \Gamma_{11}A_{1,n+1}^{(1)} & \Gamma_{11}A_{1,n+2}^{(1)} \\ \Gamma_{21}A_{n+2,n+1}^{(1)} & \Gamma_{21}A_{n+2,n+2}^{(1)} + \Gamma_{22} \end{bmatrix}^{-1} \tag{38b}$$

It can be verified that the matrices $[\tilde{\mathbf{A}}^{(r)}]$ and $[\tilde{\mathbf{B}}^{(r)}]$ in Equation (38) are independent of the value of $x_{n+1}$.

Once the weighting coefficient matrices are evaluated, the differential quadrature analogous equations of the governing differential equations (with the boundary conditions incorporated) are given by

$$([\tilde{\mathbf{A}}^{(2)}] + \alpha_1[\tilde{\mathbf{A}}^{(1)}] + \alpha_2[\mathbf{I}])\{\mathbf{Y}_1\} = \{\mathbf{f}\} - ([\tilde{\mathbf{B}}^{(2)}] + \alpha_1[\tilde{\mathbf{B}}^{(1)}])\{\boldsymbol{\beta}\} \tag{39}$$

## 7. FOURTH-ORDER EQUATIONS

Consider a fourth-order equation in the form

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + \alpha_1 \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \alpha_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha_3 \frac{\mathrm{d}y}{\mathrm{d}x} + \alpha_4 y = f(x) \quad \text{for } 0 < x < 1 \tag{40}$$

with Dirichlet and Neumann-types boundary conditions:

$$y(0) = u_1, \quad y'(0) = \theta_1, \quad y(1) = u_2, \quad y'(1) = \theta_2 \tag{41}$$

 

Since the boundary points $x=0$ and 1 have to be included, let the $n+4$ sampling grid points be arranged in ascending order as $0=x_1<x_2<\cdots<x_{n+3}<x_{n+4}=1$. The four boundary conditions can be written as

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ A^{(1)}_{1,1} & A^{(1)}_{1,2} & \cdots & A^{(1)}_{1,n+3} & A^{(1)}_{1,n+4} \\ 0 & 0 & \cdots & 0 & 1 \\ A^{(1)}_{n+4,1} & A^{(1)}_{n+4,2} & \cdots & A^{(1)}_{n+4,n+3} & A^{(1)}_{n+4,n+4} \end{bmatrix} \begin{Bmatrix} y_1 \\ \vdots \\ y_{n+4} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} \tag{42}$$

Any four unknowns from $y_1,\ldots,y_{n+4}$ can be eliminated using Equation (42). In general, the sampling grid points $x_3,\ldots,x_{n+2}$ will be chosen to establish the differential quadrature analogous equations of the governing differential equations. The remaining points $x_1=0$, $x_2, x_{n+3}$ and $x_{n+4}=1$ are treated as auxiliary sampling grid points. In this case, Equation (42) gives

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_{n+3} \\ y_{n+4} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A^{(1)}_{1,1} & A^{(1)}_{1,2} & A^{(1)}_{1,n+3} & A^{(1)}_{1,n+4} \\ 0 & 0 & 0 & 1 \\ A^{(1)}_{n+4,1} & A^{(1)}_{n+4,2} & A^{(1)}_{n+4,n+3} & A^{(1)}_{n+4,n+4} \end{bmatrix}^{-1} \left( \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ A^{(1)}_{1,3} & \cdots & A^{(1)}_{1,n+2} \\ 0 & \cdots & 0 \\ A^{(1)}_{n+4,3} & \cdots & A^{(1)}_{n+4,n+2} \end{bmatrix} \begin{Bmatrix} y_3 \\ \vdots \\ y_{n+2} \end{Bmatrix} \right) \tag{43}$$

The differential quadrature rules are then written as

$$\begin{Bmatrix} y^{(r)}_3 \\ \vdots \\ y^{(r)}_{n+2} \end{Bmatrix} = [\tilde{\mathbf{A}}^{(r)}] \begin{Bmatrix} y_3 \\ \vdots \\ y_{n+2} \end{Bmatrix} + [\tilde{\mathbf{B}}^{(r)}] \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} \tag{44}$$

where

$$[\tilde{\mathbf{A}}^{(r)}] = \begin{bmatrix} A^{(r)}_{33} & \cdots & A^{(r)}_{3,n+2} \\ \vdots & & \vdots \\ A^{(r)}_{n+2,3} & \cdots & A^{(r)}_{n+2,n+2} \end{bmatrix} - [\tilde{\mathbf{B}}^{(r)}] \begin{bmatrix} 0 & \cdots & 0 \\ A^{(1)}_{1,3} & \cdots & A^{(1)}_{1,n+2} \\ 0 & \cdots & 0 \\ A^{(1)}_{n+4,3} & \cdots & A^{(1)}_{n+4,n+2} \end{bmatrix} \tag{45a}$$

$$[\tilde{\mathbf{B}}^{(r)}] = \begin{bmatrix} A^{(r)}_{31} & A^{(r)}_{3,2} & A^{(r)}_{3,n+3} & A^{(r)}_{3,n+4} \\ \vdots & \vdots & \vdots & \vdots \\ A^{(r)}_{n+2,1} & A^{(r)}_{n+2,2} & A^{(r)}_{n+2,n+3} & A^{(r)}_{n+2,n+4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ A^{(1)}_{1,1} & A^{(1)}_{1,2} & A^{(1)}_{1,n+3} & A^{(1)}_{1,n+4} \\ 0 & 0 & 0 & 1 \\ A^{(1)}_{n+4,1} & A^{(1)}_{n+4,2} & A^{(1)}_{n+4,n+3} & A^{(1)}_{n+4,n+4} \end{bmatrix}^{-1} \tag{45b}$$

The differential quadrature rules in Equation (44) can also be obtained from the collocation method by using the Hermite functions as the trial functions [17]. Alternatively, it can also be established by inverting matrices as reported in Reference [16]. It can be seen that the determination of the trial functions satisfying the given boundary conditions is already quite complicated for the second-order equations. The formulation is even more complicated for higher-order equations with mixed boundary conditions. The present formulation therefore has an advantage of being much easier to implement and equivalent results are to be obtained.

Once the weighting coefficient matrices are evaluated, the differential quadrature analogous equations of the governing differential equations (with the boundary conditions incorporated) are given by

$$\begin{aligned}([\tilde{\mathbf{A}}^{(4)}] + \alpha_1[\tilde{\mathbf{A}}^{(3)}] + \alpha_2[\tilde{\mathbf{A}}^{(2)}] + \alpha_3[\tilde{\mathbf{A}}^{(1)}] + \alpha_4[\mathbf{I}])\{\mathbf{Y}_1\} \\ = \{\mathbf{f}\} - ([\tilde{\mathbf{B}}^{(4)}] + \alpha_1[\tilde{\mathbf{B}}^{(3)}] + \alpha_2[\tilde{\mathbf{B}}^{(2)}] + \alpha_3[\tilde{\mathbf{B}}^{(1)}])\{\boldsymbol{\beta}\}\end{aligned} \tag{46}$$

It can be verified that the matrices $[\tilde{\mathbf{A}}^{(r)}]$ and $[\tilde{\mathbf{B}}^{(r)}]$ are independent of the values of $x_2$ and $x_{n+3}$. Since $x_1 = 0$ and $x_{n+4} = 1$ are the boundary points, the remaining $n$ sampling grid points that need to be chosen are $x_3, \dots, x_{n+2}$. They can be equally spaced grid points or roots of some orthogonal polynomials. Note that $x_2$ and $x_{n+3}$ are not included in the selection. Eventually, $x_2$ and $x_{n+3}$ can be assigned some convenient values, for example,

$$x_2 = (x_1 + x_3)/2 \quad \text{and} \quad x_{n+3} = (x_{n+2} + x_{n+4})/2 \tag{47}$$

Other choices are also possible. For example, $x_2$ and $x_{n+3}$ could be

$$x_2 = (x_3 + x_4)/2 \quad \text{and} \quad x_{n+3} = (x_{n+1} + x_{n+2})/2 \tag{48}$$

It can be verified that the same modified weighting coefficient matrices in Equations (45a) and (45b) will be generated.

In conclusion, the most important sampling grid points are those used to establish the differential quadrature analogous equations of the governing differential equations. They are the essential sampling grid points. To establish the differential quadrature analogous equations of the boundary conditions, the boundary points with boundary conditions specified should be included as auxiliary sampling grid points if they are not already included in the essential sampling grid points. Additional auxiliary sample grid points are used to supply the required $n + m$ distinct sampling grid points to construct the polynomials (trial functions) of degree $n + m - 1$. In this case, the actual values of these additional auxiliary sample grid points are irrelevant.

It can also be concluded that the $\delta$ grid points are not necessary. If the boundary conditions are not apply to these $\delta$ grid points and no differential quadrature analogous equation of the governing differential equation is established at these points, the actual values for the $\delta$ grid points are irrelevant. Hence, these points need not be too close to the boundary points.

In the formulation, the differential quadrature analogous equations of the governing differential equations at any sampling grid point can be discarded and replaced by the differential quadrature analogous equations of the boundary conditions. However, it is essential that the remaining sample grid points for the differential quadrature analogous equations of the differential equations should be well spread to cover the domain under investigation. It is common that the sampling grid points on and just next to the boundaries are treated as auxiliary sampling grid points. Sometimes, the remaining sampling grid points are not well spread. As a result, some transformations have been suggested to sketch the sample grid points out to have a better coverage [15, 31, 32]. Again, under the present viewpoint, this transformation is not necessary. The essential sampling grid points should be specified directly and have a good spread initially. For example, these essential sampling grid points should be the Chebyshev–Gauss–Lobatto points without considering the auxiliary sampling grid points. Hence, future studies on the accuracy of the numerical solutions should focus on the distribution of the essential sampling grid points only, rather than which sampling grid points should be discarded [31].

## 8. NUMERICAL EXAMPLES

### *8.1. Free vibration of beams with simply supported and clamped ends*

The non-dimensional governing equation for the free vibration of a uniform beam is

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = \Omega^2 w \tag{49}$$

where $\Omega = \rho A_0 L^4 \omega^2 / EI$ is the dimensionless natural frequency, $w$, $\omega$, $A_0$, $L$, $\rho$, $E$, and $I$ are the lateral deflection, the natural frequency of free vibration, the constant cross-sectional area, the length of the beam, the mass density, the elastic modulus, and the constant area moment of inertia about the neutral axis, respectively. Since the governing equation is fourth order, two boundary conditions are needed at each end. The simply supported, clamped, and free boundary conditions can be expressed as

$$w = 0 \quad \text{and} \quad \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \quad \text{for simply supported end} \tag{50a}$$

$$w = 0 \quad \text{and} \quad \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \quad \text{for clamped end} \tag{50b}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \quad \text{and} \quad \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = 0 \quad \text{for free end} \tag{50c}$$

Consider the free vibration of a simply supported beam. Table I shows the first two dimensionless natural frequencies $\Omega_1$ and $\Omega_2$ obtained by various methods. The exact solution for $\Omega_k$ is $k^2\pi^2$. It can be seen that very accurate numerical results are obtained by using the approach proposed by Wang and Bert [9] and Malik and Bert [11] when uniform sampling grid points are used. If there are $N$ equally spaced sampling grid points, the co-ordinates of the sampling grid points are given by

$$x_i = \frac{i-1}{N-1} \quad \text{for } i = 1, \ldots, N \tag{51}$$

If the following non-uniform sampling grid points are used,

$$x_i = \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\,\pi\right)\right) \quad \text{for } i = 1, \ldots, N \tag{52}$$

the numerical results will be even more accurate. The sampling grid points given in Equation (52) are also known as the Chebyshev–Gauss–Lobatto points. After the imposition of boundary conditions, the number of unknowns reduces to $N - 2$.

 

Table I. Comparison of results for free vibration of a simply supported beam.

| Method | Type of grid points | No. of grid points | No. of unknowns | $\Omega_1$ | $\Omega_2$ | Relative error in $\Omega_1$ | Relative error in $\Omega_2$ |
|---|---|---|---|---|---|---|---|
| Exact | — | — | — | 9.8696 | 39.4784 | — | — |
| Modify weighting coefficient matrices as in References [9, 11] | Uniform Equation (51) | 7 | 5 | 9.8675 | 41.6495 | −2.15E-04 | 5.50E-02 |
| | | 8 | 6 | 9.8683 | 39.2411 | −1.29E-04 | −6.01E-03 |
| Modify weighting coefficient matrices as presented in this paper | Uniform+aux Equation (53) | 7 | 3 | 9.9165 | 35.0542 | 4.75E-03 | −1.12E-01 |
| | | 8 | 4 | 9.8948 | 41.3037 | 2.56E-03 | 4.62E-02 |
| | | 9 | 5 | 9.8686 | 40.4094 | −1.06E-04 | 2.36E-02 |
| | | 10 | 6 | 9.8690 | 39.3634 | −6.40E-05 | −2.91E-03 |
| Modify weighting coefficient matrices as presented in this paper | Uniform Equation (51) | 7 | 3 | 9.9591 | 31.1769 | 9.07E-03 | −2.10E-01 |
| | | 8 | 4 | 9.9364 | 44.9208 | 6.76E-03 | 1.38E-01 |
| | | 9 | 5 | 9.8669 | 42.8479 | −2.69E-04 | 8.54E-02 |
| | | 10 | 6 | 9.8676 | 39.1580 | −2.01E-04 | −8.12E-03 |
| Modify weighting coefficient matrices as in References [9, 11] | Non-uniform Equation (52) | 7 | 5 | 9.8697 | 39.5133 | 8.01E-06 | 8.83E-04 |
| | | 8 | 6 | 9.8696 | 39.4778 | 2.73E-06 | −1.60E-05 |
| Modify weighting coefficient matrices as presented in this paper | Non-uniform+aux Equation (54) | 7 | 3 | 9.8449 | 45.7662 | −2.51E-03 | 1.59E-01 |
| | | 8 | 4 | 9.8688 | 38.8385 | −7.93E-05 | −1.62E-02 |
| | | 9 | 5 | 9.8696 | 39.5292 | 3.51E-06 | 1.29E-03 |
| | | 10 | 6 | 9.8696 | 39.4768 | 1.36E-06 | −4.13E-05 |
| Modify weighting coefficient matrices as presented in this paper | Non-uniform Equation (52) | 7 | 3 | 9.9165 | 35.0542 | 4.75E-03 | −1.12E-01 |
| | | 8 | 4 | 9.8913 | 40.9341 | 2.19E-03 | 3.69E-02 |
| | | 9 | 5 | 9.8689 | 40.0744 | −7.28E-05 | 1.51E-02 |
| | | 10 | 6 | 9.8693 | 39.4174 | −3.41E-05 | −1.55E-03 |

For the present method, 4 auxiliary sampling grid points are required. If there are $N$ sampling grid points, then $N-4$ of them will be the essential sampling grid points with the differential quadrature analogous equations of the governing differential equations established. The $N-4$ sampling grid points may be uniform or non-uniform. The co-ordinates are given by

$$x_i = \frac{i}{n+1}, \quad i=1,\ldots,n \quad \text{for uniform sampling grid points} \tag{53}$$

$$x_i = \frac{1}{2}\left(1-\cos\left(\frac{i}{n+1}\,\pi\right)\right), \quad i=1,\ldots,n \quad \text{for non-uniform sampling grid points} \tag{54}$$

where $n=N-4$. The two end points could be included as $x_{n+1}$ and $x_{n+2}$ (i.e. $x_{n+1}=0$ and $x_{n+2}=1$). Another 2 sampling grid points, $x_{n+3}$ and $x_{n+4}$, can be arbitrary. In the $\delta$-technique, $x_{n+3}$ and $x_{n+4}$ are close to the boundary points (i.e. $x_{n+3}=\delta$ and $x_{n+4}=1-\delta$). In the present calculation, $x_{n+3}=x_1/2$ and $x_{n+4}=(x_n+1)/2$.

 

Table II. Comparison of results for free vibration of a clamped–simply supported beam.

| Method | Type of grid points | No. of grid points | No. of unknowns | $\Omega_1$ | $\Omega_2$ | Relative error in $\Omega_1$ | Relative error in $\Omega_2$ |
|---|---|---|---|---|---|---|---|
| Exact | — | — | — | 15.4182 | 49.9649 | — | — |
| Modify weighting coefficient matrices as in References [9, 11] | Uniform Equation (51) | 7 | 5 | 15.3383 | 60.2466 | −5.18E-03 | 2.06E-01 |
| | | 8 | 6 | 15.4019 | 47.6015 | −1.06E-03 | −4.73E-02 |
| Modify weighting coefficient matrices as in References [9, 11] | Non-uniform Equation (52) | 7 | 5 | 15.4114 | 50.3658 | −4.42E-04 | −8.02E-03 |
| | | 8 | 6 | 15.4173 | 49.7935 | −5.78E-05 | −3.43E-03 |
| Modify weighting coefficient matrices as presented in this paper | Non-uniform+aux Equation (54) | 7 | 3 | 15.3592 | 58.7377 | −3.83E-03 | 1.76E-01 |
| | | 8 | 4 | 15.4221 | 48.9954 | 2.50E-04 | −1.94E-02 |
| | | 9 | 5 | 15.4180 | 50.1467 | −1.24E-05 | 3.64E-03 |
| | | 10 | 6 | 15.4182 | 49.9504 | 2.86E-06 | −2.90E-04 |

From Table I, it can be seen that for the same number of sampling grid points, the results obtained by the method proposed by Wang and Bert [9] and Malik and Bert [11] are more accurate than the present method. However, it should be noted that the number of unknowns for their method is $N-2$ while it is $N-4$ for the present method. If the same number of unknowns is considered, from Table I, it can be seen that the results obtained by the two methods are comparable for both uniform and non-uniform sampling grid points.

Table I also shows the results obtained by the conventional approach, i.e. using the sampling grid points in Equations (51) and (52) for both the essential and auxiliary sampling grid points. It can be seen that the results are not very good, as the essential sampling grid points are not well spread.

The numerical results for the clamped–simply supported beam and the clamped–clamped beam are also considered. Tables II and III show the numerical results obtained by using various methods. It can be seen that the same conclusion can be drawn. Hence, in terms of accuracy, the present approach to modify the weighting coefficient matrices is as good as the previous method presented in References [9, 11].

However, it should be noted that the numerical results obtained by the methods presented in References [9, 11] need careful interpretation. For clamped–clamped beam, it can be verified that a zero eigenvalue is always presented. Hence, the stiffness matrix is singular and cannot be used to solve static problems. As a result, the procedure to impose the boundary conditions into the weighting coefficient matrices has to be reviewed carefully again. On the other hand, the present procedure to impose the boundary conditions into the modified weighting coefficient matrices does not give zero eigenvalue, as the present method is equivalent to the collocation method. This is an important improvement over the existing approaches on how the weighting coefficient matrices should be modified.

 

Table III. Comparison of results for free vibration of a clamped–clamped beam.

| Method | Type of grid points | No. of grid points | No. of unknowns | $\Omega_1$ | $\Omega_2$ | Relative error in $\Omega_1$ | Relative error in $\Omega_2$ |
|---|---|---|---|---|---|---|---|
| Exact | — | — | — | 22.3733 | 61.6728 | — | — |
| Modify weighting coefficient matrices as in References [9, 11] (with the first zero eigenvalue ignored) | Uniform Equation (51) | 7 | 5 | No real values | | — | — |
| | | 8 | 6 | 22.2891 | No value | −3.76E-03 | — |
| Modify weighting coefficient matrices as in References [9, 11] (with the first zero eigenvalue ignored) | Non-uniform Equation (52) | 7 | 5 | 23.5535 | 63.0943 | 5.27E-02 | 2.30E-02 |
| | | 8 | 6 | 22.3676 | 66.2463 | −2.54E-04 | 7.42E-02 |
| Modify weighting coefficient matrices as presented in this paper | Non-uniform+aux Equation (54) | 7 | 3 | 21.9677 | 87.6356 | −1.81E-02 | 4.21E-01 |
| | | 8 | 4 | 22.4442 | 58.8049 | 3.17E-03 | −4.65E-02 |
| | | 9 | 5 | 22.3707 | 62.6132 | −1.17E-04 | 1.52E-02 |
| | | 10 | 6 | 22.3733 | 61.5867 | 9.01E-08 | −1.40E-03 |

### *8.2. Free vibration of a cantilever beam*

There can be some interesting observation made from the numerical results presented in Reference [1]. From Table III in Reference [1], it can be seen that the numerical results for $N=7$ for Types II and III sampling grid points are the same. The sampling grid points for Types II and III with $N=7$ are given by

$$\text{Type II:}\ x_1=0,\ x_2=\delta,\ x_3=\tfrac{1}{4},\ x_4=\tfrac{1}{2},\ x_5=\tfrac{3}{4},\ x_6=1-\delta,\ x_7=1 \tag{55a}$$

$$\text{Type III:}\ x_1=0,\ x_2=\tfrac{1}{2}-\tfrac{\sqrt{3}}{4},\ x_3=\tfrac{1}{4},\ x_4=\tfrac{1}{2},\ x_5=\tfrac{3}{4},\ x_6=\tfrac{1}{2}+\tfrac{\sqrt{3}}{4},\ x_7=1 \tag{55b}$$

It can be seen that, even though $x_2$ and $x_6$ are different in these two cases, they give the same numerical results since $x_2$ and $x_6$ are the auxiliary sampling grid points only. This confirms the present argument that the actual values for the auxiliary sampling grid points are irrelevant.

If the essential and auxiliary sampling grid points can be considered separately, the choice of the sampling grid points could be more flexible. For example, the Legendre–Gauss points can be used for the essential sampling grid points only. The auxiliary sampling grid points are then decided later on. Table IV shows that the numerical results given by the Legendre–Gauss grid points could be better than the results given by the unequally spaced sampling points with adjacent $\delta$-grid points (Type IV in Reference [1]), i.e.

$$x_1=0,\ x_2=\delta,\ x_{N-1}=1-\delta,\ x_N=1,\ x_i=\frac{1}{2}\left(1-\cos\left(\frac{i-1}{N-3}\pi\right)\right),\quad i=3,\ldots,N-2 \tag{56}$$

 

Table IV. Comparison of results for free vibration of a cantilever beam.

| Method | No. of grid points | No. of unknowns | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | Relative error in $\Omega_1$ | Relative error in $\Omega_2$ | Relative error in $\Omega_3$ |
|---|---|---|---|---|---|---|---|---|
| Exact | — | — | 3.516015 | 22.034492 | 61.697214 | — | — | — |
| Type IV from | 7 | 3 | 3.508752 | — | — | −2.07E-03 | — | — |
| Table 3 in Reference [1] | 8 | 4 | 3.516708 | 21.922707 | — | 1.97E-04 | −5.07E-03 | — |
| Equation (56) | 9 | 5 | 3.516088 | 22.058830 | 61.732975 | 2.07E-05 | 1.10E-03 | 5.80E-04 |
| | 10 | 6 | 3.516017 | 22.041802 | 61.834629 | 4.84E-07 | 3.32E-04 | 2.23E-03 |
| Also the present | 11 | 7 | 3.516015 | 22.034379 | 61.780091 | −8.53E-08 | −5.11E-06 | 1.34E-03 |
| method with | 12 | 8 | 3.516015 | 22.034364 | 61.689815 | −8.53E-08 | −5.79E-06 | −1.20E-04 |
| Equation (54) | 13 | 9 | 3.516015 | 22.034498 | 61.693836 | −8.53E-08 | 2.90E-07 | −5.48E-05 |
| The present | 7 | 3 | 3.517682 | — | — | 4.74E-04 | — | — |
| method with | 8 | 4 | 3.516009 | 22.201720 | — | −1.72E-06 | 7.59E-03 | — |
| Legendre–Gauss | 9 | 5 | 3.516016 | 22.027964 | 63.712564 | 6.07E-08 | −2.96E-04 | 3.27E-02 |
| sampling grid | 10 | 6 | 3.516015 | 22.034954 | 61.514352 | −9.09E-09 | 2.10E-05 | −2.96E-03 |
| points | 11 | 7 | 3.516015 | 22.034482 | 61.723048 | −8.96E-09 | −4.38E-07 | 4.19E-04 |
| | 12 | 8 | 3.516015 | 22.034492 | 61.695856 | −8.96E-09 | 1.33E-08 | −2.20E-05 |
| | 13 | 9 | 3.516015 | 22.034492 | 61.697313 | −8.96E-09 | −1.80E-09 | 1.59E-06 |
| Modified weighting | 7 | 6 | 3.515695 | 21.596681 | No value | −9.11E-05 | −1.99E-02 | — |
| coefficients matrices | 8 | 7 | 3.516005 | 21.896604 | 54.886288 | −2.99E-06 | −6.26E-03 | −1.10E-01 |
| [9, 11] with | 9 | 8 | 3.516017 | 22.039554 | 58.845985 | 5.17E-07 | 2.30E-04 | −4.62E-02 |
| Equation (51) | 10 | 9 | 3.516015 | 22.039173 | 62.387781 | −3.83E-08 | 2.12E-04 | 1.12E-02 |
| Modified weighting | 7 | 6 | 3.515983 | 21.985978 | 63.412011 | −9.29E-06 | −2.20E-03 | 2.78E-02 |
| coefficients matrices | 8 | 7 | 3.516014 | 22.025234 | 60.871860 | −2.44E-07 | −4.20E-04 | −1.34E-02 |
| [9, 11] with | 9 | 8 | 3.516015 | 22.034618 | 61.558912 | 1.51E-08 | 5.75E-06 | −2.24E-03 |
| Equation (52) | 10 | 9 | 3.516015 | 22.034622 | 61.710080 | −9.77E-09 | 5.91E-06 | 2.09E-04 |

Table V. Comparison of results for free vibration of a free–free beam by using various sampling grid points.

| Type of grid points | No. of grid points | No. of unknowns | $\Omega_3$ | $\Omega_4$ | Relative error in $\Omega_3$ | Relative error in $\Omega_4$ |
|---|---|---|---|---|---|---|
| Exact solution | — | — | 22.3733 | 61.6728 | — | — |
| Uniform | 9 | 5 | 22.9391 | 48.2647 | 2.53E-02 | −2.17E-01 |
| Equation (51) | 10 | 6 | 22.8291 | 78.2793 | 2.04E-02 | 2.69E-01 |
| Uniform+sketch | 9 | 5 | 22.6646 | 53.1711 | 1.30E-02 | −1.38E-01 |
| Equations (51) and (57) | 10 | 6 | 22.5512 | 66.3655 | 7.95E-03 | 7.61E-02 |
| Uniform+aux | 9 | 5 | 22.7030 | 52.2343 | 1.47E-02 | −1.53E-01 |
| Equation (53) | 10 | 6 | 22.6058 | 68.4660 | 1.04E-02 | 1.10E-01 |
| Non-uniform | 9 | 5 | 22.6347 | 53.8801 | 1.17E-02 | −1.26E-01 |
| Equation (52) | 10 | 6 | 22.5266 | 65.5865 | 6.85E-03 | 6.35E-02 |
| Non-uniform+sketch | 9 | 5 | 22.3904 | 62.8837 | 7.64E-04 | 1.96E-02 |
| Equations (52) and (57) | 10 | 6 | 22.3878 | 61.5654 | 6.48E-04 | −1.74E-03 |
| Non-uniform+aux | 9 | 5 | 22.4143 | 60.9952 | 1.83E-03 | −1.10E-02 |
| Equation (54) | 10 | 6 | 22.3905 | 62.0076 | 7.70E-04 | 5.43E-03 |
| Non-uniform+aux | 9 | 5 | 22.3665 | 63.5317 | −3.04E-04 | 3.01E-02 |
| Legendre points | 10 | 6 | 22.3738 | 61.4931 | 2.49E-05 | −2.91E-03 |

*Note*: The first two zero eigenvalues are omitted.

Note that even though the $\delta$-grid points are used, the boundary conditions are applied at the boundary points and not at the $\delta$-grid points (see Example 3 in Reference [1] for more details). The same numerical results can be obtained by the present approach if the essential sampling grid points in Equation (54) are used.

Table IV also shows the results obtained by the using the modified weighting coefficient matrices presented in Reference [11] with uniform and non-uniform sampling grid points given by Equations (51) and (52). It can be seen that, for the same number of unknowns, the present method using the Legendre–Gauss points gives better results. Hence, the ability to separate the essential sampling grid points from the auxiliary sampling grid points gives more flexibility in choosing the sampling grid points and, hence, could give more accurate results.

### *8.3. Free vibration of a free–free beam*

It has been suggested that the remaining sampling grid points may need to be stretched towards the boundaries in order to get better numerical results [15, 31, 32]. For example, it was recommended that the sampling grid points $x_i$ should be transformed by using the formula

$$(1-\alpha)(3x_i^2 - 2x_i^3) + \alpha x_i \tag{57}$$

This is especially important for free–free beams. It is recommended that $\alpha = 0$ should be used. The numerical results obtained by using various methods are shown in Table V. It can be seen that the numerical results are improved by using the transformation in Equation (57) to stretch the sampling grid points. The main function is to give a better coverage of the sampling grid points. It can be seen that, in the present formulation, comparable results can be obtained and there is no need to apply any transformation since the sampling grid points are specified directly. In fact, by specifying the Legendre–Gauss sampling grid points as the essential sampling grid points, more accurate results can be obtained. The present formulation therefore is more flexible in choosing the sampling grid points. It is advocated in this paper that the main focus should be on the choice of the essential sampling grid points.

### 8.4. *Deflection of thin rectangular plates*

The normalized governing equation for a thin rectangular plate is

$$\frac{\partial^4 W}{\partial X^4} + 2\lambda^2 \frac{\partial^4 W}{\partial X^2 \partial Y^2} + \lambda^4 \frac{\partial^4 W}{\partial Y^4} = \frac{pa^4}{D} \tag{58}$$

where $W$, $X = x/a$, and $Y = y/b$ are the normalized dimensionless deflection and co-ordinates, respectively, $a$ and $b$ are the length and the width of the rectangular plate, respectively, $\lambda = a/b$ is the aspect ratio, $D = Eh^3/(12(1-\upsilon^2))$ is the flexural rigidity, $E$, $\upsilon$, $h$, and $p$ are the Young's modulus, Poisson's ratio, plate thickness, and the lateral load on the plate, respectively.

The boundary conditions for a plate with all four edges clamped are

$$\begin{gathered} W(X,0) = W(X,1) = W(0,Y) = W(1,Y) = 0 \\ \frac{\partial W}{\partial Y}(X,0) = \frac{\partial W}{\partial Y}(X,0) = \frac{\partial W}{\partial X}(0,Y) = \frac{\partial W}{\partial X}(1,Y) = 0 \end{gathered} \tag{59}$$

and the boundary conditions for a plate with all four edges simply supported are

$$\begin{gathered} W(X,0) = W(X,1) = W(0,Y) = W(1,Y) = 0 \\ \frac{\partial^2 W}{\partial Y^2}(X,0) = \frac{\partial^2 W}{\partial Y^2}(X,0) = \frac{\partial^2 W}{\partial X^2}(0,Y) = \frac{\partial^2 W}{\partial X^2}(1,Y) = 0 \end{gathered} \tag{60}$$

Many researchers have used the differential quadrature method to find the static deflection under various types of loading and boundary conditions [6, 7, 14, 15]. In the present study, the main concern is the influence of the distribution of the sampling grid points. Table VI shows the numerical results obtained by using various methods when $pa^4/D = 1000$. $W_{\mathrm{ssss}}$ and $W_{\mathrm{cccc}}$ are used to denote the calculated central deflections of the plate with all four edges simply supported and clamped, respectively. It can be seen that more accurate results are obtained if the essential sampling grid points are chosen from the Legendre or Chebyshev–Gauss–Lobatto points. The auxiliary sampling grid points should not be included when considering the choice of the sample grid points.

As mentioned in Reference [11], some boundary conditions may involve mixed derivatives, for example, bending moment for an-isotopic plates or composite plates, or the $\partial^3 W/\partial^2 X \partial Y$ and $\partial^3 W/\partial X \partial^2 Y$ terms for zero effective shear force. In these cases, the boundary conditions cannot be incorporated in the weighting coefficient matrices directly since the weighting coefficient matrices involve only one spatial direction. This situation can still be handled by constructing the differential quadrature analogous equations of the appropriate boundary conditions, as in Reference [15]. For skewed or composite plates, other special techniques for the differential quadrature method are available [45–49].

Table VI. Comparison of results for static deflection of square plate by using various sampling grid points ($pa^2/D = 1000$).

| Type of grid points | No. of grid points | No. of unknowns | $W_{ssss}$ | $W_{cccc}$ | Relative error in $W_{ssss}$ | Relative error in $W_{cccc}$ |
|---|---|---|---|---|---|---|
| Exact solution | — | — | 4.0624 | 1.2653 | — | — |
| Uniform | 7×7 | 9 | 3.9560 | 1.2344 | −2.62E-02 | −2.45E-02 |
| Equation (51) | 9×9 | 25 | 4.0554 | 1.2665 | −1.72E-03 | 9.04E-04 |
| Uniform+aux | 7×7 | 9 | 4.0037 | 1.2568 | −1.44E-02 | −6.72E-03 |
| Equation (53) | 9×9 | 25 | 4.0599 | 1.2656 | −6.13E-04 | 2.01E-04 |
| Non-uniform | 7×7 | 9 | 4.0037 | 1.2568 | −1.44E-02 | −6.72E-03 |
| Equation (52) | 9×9 | 25 | 4.0607 | 1.2655 | −4.02E-04 | 9.20E-05 |
| Non-uniform+aux | 7×7 | 9 | 4.0966 | 1.2826 | 8.44E-03 | 1.36E-02 |
| Equation (54) | 9×9 | 25 | 4.0626 | 1.2655 | 5.33E-05 | 1.23E-04 |
| Non-uniform+aux | 7×7 | 9 | 4.1300 | 1.2857 | 1.66E-02 | 1.61E-02 |
| Legendre points | 9×9 | 25 | 4.0625 | 1.2682 | 3.31E-05 | 2.25E-03 |

### *8.5. Isothermal reactor with axial mixing*

Consider the steady-state solution of an isothermal reactor with axial mixing. The governing equation is given by [12]

$$\frac{1}{Pe}\frac{\mathrm{d}^2p}{\mathrm{d}x^2} - \frac{\mathrm{d}p}{\mathrm{d}x} - rp^2 = 0 \quad \text{for } 0 \leqslant x \leqslant L \tag{61}$$

with the Cauchy and Neumann boundary conditions, respectively

$$p - \frac{1}{Pe}\frac{\mathrm{d}p}{\mathrm{d}x} = p^* \quad \text{at } x = 0 \tag{62a}$$

and

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 0 \quad \text{at } x = L \tag{62b}$$

where $x$ is the distance, $L$ is the reactor length, $p$ is the reactor partial pressure, $p^*$ is the entrance partial pressure of the reactor, $Pe$ is the Peclet number, and $r$ is the reactor rate number. Let $X = x/L$ and $P = p/p^*$ denote the non-dimensional spatial and pressure variables. Equation (61) can then be rewritten as

$$\frac{1}{PeL}\frac{\mathrm{d}^2P}{\mathrm{d}X^2} - \frac{\mathrm{d}P}{\mathrm{d}X} - rLp^*P^2 = 0 \quad \text{for } 0 \leqslant X \leqslant 1 \tag{63}$$

with boundary conditions

$$P - \frac{1}{PeL}\frac{\mathrm{d}P}{\mathrm{d}X} = 1 \text{ at } X = 0 \quad \text{and} \quad \frac{\mathrm{d}P}{\mathrm{d}X} = 0 \text{ at } X = 1 \tag{64}$$

 

The differential quadrature analog equation for Equation (63) at $x = x_i$ can be written as

$$\frac{1}{PeL} P_i^{(2)} - P_i^{(1)} - rLp^* P_i^2 = 0 \quad \text{for } 1 \leqslant i \leqslant n \tag{65}$$

where

$$P_k = P(X_k), \quad P_k^{(r)} = \left.\frac{\mathrm{d}^r P}{\mathrm{d}X^r}\right|_{X=X_k} \tag{66}$$

Let

$$\{\mathbf{P}\} = \begin{Bmatrix} P_1 \\ \vdots \\ P_n \end{Bmatrix}, \quad \{\mathbf{P}^{(1)}\} = \begin{Bmatrix} P_1^{(1)} \\ \vdots \\ P_n^{(1)} \end{Bmatrix}, \quad \{\mathbf{P}^{(2)}\} = \begin{Bmatrix} P_1^{(2)} \\ \vdots \\ P_n^{(2)} \end{Bmatrix} \tag{67}$$

Using the differential quadrature rule, $\{\mathbf{P}^{(1)}\}$ is related to $\{\mathbf{P}\}$ by

$$\{\mathbf{P}^{(1)}\} = [\mathbf{A}^{(1)}]\{\mathbf{P}\} \tag{68}$$

To impose the boundary conditions, Civan [12] suggested that $P_1^{(1)}$ and $P_n^{(1)}$ in Equation (68) should be replaced by

$$P_1^{(1)} = PeL(P_1 - 1) \quad \text{and} \quad P_n^{(1)} = 0 \tag{69}$$

Therefore, Equation (68) is modified to

$$\begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ \vdots \\ P_{n-1}^{(1)} \\ P_n^{(1)} \end{Bmatrix} = \begin{bmatrix} PeL & 0 & \cdots & 0 & 0 \\ A_{21}^{(1)} & A_{22}^{(1)} & \cdots & A_{2,n-1}^{(1)} & A_{2,n}^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ A_{n-1,1}^{(1)} & A_{n-1,2}^{(1)} & \cdots & A_{n-1,n-1}^{(1)} & A_{n-1,n}^{(1)} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ P_n \end{Bmatrix} - \begin{Bmatrix} PeL \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix} \tag{70}$$

The differential quadrature rule for the second derivative $\{\mathbf{P}^{(2)}\}$ is related to $\{\mathbf{P}^{(1)}\}$ by

$$\{\mathbf{P}^{(2)}\} = [\mathbf{A}^{(1)}]\{\mathbf{P}^{(1)}\} \tag{71}$$

Note that, in general, $\{\mathbf{P}^{(2)}\} \neq [\mathbf{A}^{(1)}]^2\{\mathbf{P}\}$. From Equations (70) and (71), $\{\mathbf{P}^{(1)}\}$ and $\{\mathbf{P}^{(2)}\}$ can be expressed in terms of $\{\mathbf{P}\}$. Hence, the $n$ unknowns $P_1, P_2, \ldots, P_n$ in $\{\mathbf{P}\}$ can be solved from the $n$ differential quadrature analogous equations in Equation (65) at $X_1, \ldots, X_n$.

The numerical solutions with $p^* = 0.07$, $Pe = 2$, $r = 1$, and $L = 48$ using 5 equal intervals ($n = 6$) and 10 equal intervals ($n = 11$) are shown in Table VII. The finite difference solutions given by Lee [50] are also included for comparison. Civan [12] remarked that the quadrature solutions agreed reasonably well except for the solution at the outlet end of the reactor ($x = L$ or $X = 1$). It is found that the solution at the outlet is still not very accurate even when 20 equal intervals ($n = 21$) are used. A close examination shows that the numerical solutions

Table VII. Comparison of the numerical solutions for isothermal reactor with axial mixing by various methods.

| | Pressure $p$ $(=P^{*}0.07)$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | Lee [50] | Civan [12] | | | Present | | |
| $X$ $(=x/L)$ | FDM | 5 Equal intervals | 10 Equal intervals | 20 Equal intervals | 5 Equal intervals | 10 Equal intervals | 5 Unequal intervals* |
| 0.0 | 0.068 | 0.067837 | 0.067772 | 0.067781 | 0.068197 | 0.067798 | 0.068025 |
| 0.2 | 0.042 | 0.039726 | 0.043746 | 0.042647 | 0.043170 | 0.041797 | 0.041434 |
| 0.4 | 0.030 | 0.029378 | 0.031126 | 0.030552 | 0.030727 | 0.030109 | 0.029798 |
| 0.6 | 0.023 | 0.022758 | 0.024105 | 0.023759 | 0.023913 | 0.023489 | 0.024012 |
| 0.8 | 0.018 | 0.019311 | 0.019631 | 0.019420 | 0.019453 | 0.019239 | 0.019281 |
| 1.0 | 0.016 | 0.007580 | 0.010482 | 0.014151 | 0.017431 | 0.016668 | 0.016416 |
| Error at $X=0$ $(e_0)$ | — | −0.599 | 2.320 | 2.416 | 0 | 0 | 0 |
| Error at $X=1$ $(e_1)$ | — | −1.949 | −3.264 | −2.874 | 0 | 0 | 0 |

*Note*:*The Legendre–Gauss points are used for the unequal intervals.

do not reflect the boundary conditions correctly. Let the error at the two ends $e_0$ and $e_1$ be defined as

$$e_0 = \sum_{j=1}^{n} A_{1j}^{(1)} P_j - PeL(P_1 - 1) \tag{72a}$$

$$e_1 = \sum_{j=1}^{n} A_{nj}^{(1)} P_j - 0 \tag{72b}$$

If the boundary conditions are correctly imposed, both $e_0$ and $e_1$ should be zero. Table VII shows that $e_0$ and $e_1$ are not zero when the weighting coefficient matrix are modified to incorporate the boundary conditions as in Equation (70).

If the boundary conditions are imposed by the present method, the different quadrature analog equations for the boundary conditions are

$$\sum_{j=1}^{n} A_{1j}^{(1)} P_j - PeLP_1 = -PeL \tag{73a}$$

$$\sum_{j=1}^{n} A_{nj}^{(1)} P_j - 0 = 0 \tag{73b}$$

Using Equation (73), $P_1$ and $P_n$ can be expressed in terms of $P_2, \ldots, P_{n-1}$. $P_2^{(r)}, \ldots, P_{n-1}^{(r)}$ can then be expressed in terms of $P_2, \ldots, P_{n-1}$ and the boundary conditions as

$$\begin{Bmatrix} P_2^{(r)} \\ \vdots \\ P_{n-1}^{(r)} \end{Bmatrix} = [\tilde{\mathbf{A}}^{(r)}] \begin{Bmatrix} P_2 \\ \vdots \\ P_{n-1} \end{Bmatrix} + [\tilde{\mathbf{B}}^{(r)}] \begin{Bmatrix} -PeL \\ 0 \end{Bmatrix} \tag{74}$$

 

where

$$[\tilde{\mathbf{A}}^{(r)}] = \begin{bmatrix} A_{22}^{(r)} & \cdots & A_{2,n-1}^{(r)} \\ \vdots & & \vdots \\ A_{n-1,2}^{(r)} & \cdots & A_{n-1,n-1}^{(r)} \end{bmatrix} - [\tilde{\mathbf{B}}^{(r)}] \begin{bmatrix} A_{12}^{(r)} & \cdots & A_{1,n-1}^{(r)} \\ A_{n2}^{(r)} & \cdots & A_{n,n-1}^{(r)} \end{bmatrix} \tag{75a}$$

$$[\tilde{\mathbf{B}}^{(r)}] = \begin{bmatrix} A_{21}^{(r)} & A_{2n}^{(r)} \\ \vdots & \vdots \\ A_{n-1,1}^{(r)} & A_{n-1,n}^{(r)} \end{bmatrix} \begin{bmatrix} A_{11}^{(r)} - PeL & A_{1n}^{(r)} \\ A_{n1}^{(r)} & A_{nn}^{(r)} \end{bmatrix}^{-1} \tag{75b}$$

$[\tilde{\mathbf{A}}^{(r)}]$ is the modified weighting coefficient matrix and $[\tilde{\mathbf{B}}^{(r)}]$ is the weighting coefficient matrix related to the given boundary conditions.

The $n-2$ differential quadrature analog equations for Equation (64) at $X_2,\ldots,X_{n-1}$ are given by

$$\frac{1}{PeL} P_i^{(2)} - P_i^{(1)} - rLp^* P_i^2 = 0, \quad 2 \leqslant i \leqslant n-1 \tag{76}$$

Using Equation (74), $P_2,\ldots,P_{n-1}$ can be solved. The numerical results given by using 5 equal intervals ($n=6$) and 10 equal intervals ($n=11$) are shown in Table VII. It can be seen that the present numerical solutions give accurate solution at the outlet end of the reactor ($x=L$ or $X=1$) even when $n=6$. It can be checked that $e_0$ and $e_1$ are zero because of Equation (73). Furthermore, the equations that need to be solved are fewer (only $n-2$) for the present method. It can be seen from Table VII that if the Legendre–Gauss points are used, more accurate results can be obtained even when $n=6$.

## 9. CONCLUSIONS

In this paper, a more complete methodology to impose the given boundary conditions by modifying the weighting coefficient matrices is presented. The boundary conditions are satisfied exactly by the interpolated solutions. The following findings are observed.

(1) The modified weighting coefficient matrices can be calculated easily. The present algorithms would be equivalent to the collocation method employing trial functions that satisfy the boundary conditions exactly if the same essential sampling grid points are used. However, the present method saves the trouble in constructing the trial functions that satisfy the given boundary conditions.
(2) It is found that the numerical results only depend on the essential sampling grid points (where the differential quadrature analogous equations of the governing differential equations are established). Hence, only the essential sampling grid points should be chosen carefully. The auxiliary sampling grid points can be arbitrary as long as they do not create numerical stability problems in evaluating the weighting coefficients. In addition, the boundary points should be included in either the essential or auxiliary

 

sampling grid points to facilitate the construction of the differential quadrature analogous equations of boundary conditions.

(3) The numerical results will also be equivalent to the conventional differential quadrature method by dropping the differential quadrature analogous equations of the governing differential equations at the auxiliary sampling grid points and replacing them with the differential quadrature analogous equations of the boundary conditions.

(4) As the present method is equivalent to the collocation method, the derived matrices do not have extra singularity. As a result, there will be no extra zero eigenvalues.

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