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SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

CONTINUUM MECHANICS

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BY

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Preface

Because of its emphasis on basic concepts and fundamental principles, Continuum Mechanics has an important role in modern engineering and technology. Several undergraduate courses which utilize the continuum concept and its dependent theories in the training of engineers and scientists are well established in today's curricula and their number continues to grow. Graduate programs in Mechanics and associated areas have long recognized the value of a substantial exposure to the subject. This book has been written in an attempt to assist both undergraduate and first year graduate students in understanding the fundamental principles of continuum theory. By including a number of solved problems in each chapter of the book, it is further hoped that the student will be able to develop his skill in solving problems in both continuum theory and its related fields of application.

In the arrangement and development of the subject matter a sufficient degree of continuity is provided so that the book may be suitable as a text for an introductory course in Continuum Mechanics. Otherwise, the book should prove especially useful as a supplementary reference for a number of courses for which continuum methods provide the basic structure. Thus courses in the areas of Strength of Materials, Fluid Mechanics, Elasticity, Plasticity and Viscoelasticity relate closely to the substance of the book and may very well draw upon its contents.

Throughout most of the book the important equations and fundamental relationships are presented in both the indicial or "tensor" notation and the classical symbolic or "vector" notation. This affords the student the opportunity to compare equivalent expressions and to gain some familiarity with each notation. Only Cartesian tensors are employed in the text because it is intended as an introductory volume and since the essence of much of the theory can be achieved in this context.

The work is essentially divided into two parts. The first five chapters deal with the basic continuum theory while the final four chapters cover certain portions of specific areas of application. Following an initial chapter on the mathematics relevant to the study, the theory portion contains additional chapters on the Analysis of Stress, Deformation and Strain, Motion and Flow, and Fundamental Continuum Laws. Applications are treated in the final four chapters on Elasticity, Fluids, Plasticity and Viscoelasticity. At the end of each chapter a collection of solved problems together with several exercises for the student serve to illustrate and reinforce the ideas presented in the text.

The author acknowledges his indebtedness to many persons and wishes to express his gratitude to all for their help. Special thanks are due the following: to my colleagues, Professors W. A. Bradley, L. E. Malvern, D. H. Y. Yen, J. F. Foss and G. LaPalm each of whom read various chapters of the text and made valuable suggestions for improvement; to Professor D. J. Montgomery for his support and assistance in a great many ways; to Dr. Richard Hartung of the Lockheed Research Laboratory, Palo Alto, California, who read the preliminary version of the manuscript and gave numerous helpful suggestions; to Professor M. C. Stippes, University of Illinois, for his invaluable comments and suggestions; to Mrs. Thelma Liszewski for the care and patience she displayed in typing the manuscript; to Mr. Daniel Schaum and Mr. Nicola Monti for their continuing interest and guidance throughout the work. The author also wishes to express thanks to his wife and children for their encouragement during the writing of the book.

Michigan State University
June 1970

GEORGE E. MASE

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CONTENTS

Chapter 1

Mathematical Foundations

1.1 TENSORS AND CONTINUUM MECHANICS

Continuum mechanics deals with physical quantities which are independent of any particular coordinate system that may be used to describe them. At the same time, these physical quantities are very often specified most conveniently by referring to an appropriate system of coordinates. Mathematically, such quantities are represented by *tensors*.

As a mathematical entity, a tensor has an existence independent of any coordinate system. Yet it may be specified in a particular coordinate system by a certain set of quantities, known as its *components*. Specifying the components of a tensor in one coordinate system determines the components in any other system. Indeed, the *law of transformation* of the components of a tensor is used here as a means for defining the tensor. Precise statements of the definitions of various kinds of tensors are given at the point of their introduction in the material that follows.

The physical laws of continuum mechanics are expressed by tensor equations. Because tensor transformations are linear and homogeneous, such tensor equations, if they are valid in one coordinate system, are valid in any other coordinate system. This *invariance* of tensor equations under a coordinate transformation is one of the principal reasons for the usefulness of tensor methods in continuum mechanics.

1.2 GENERAL TENSORS. CARTESIAN TENSORS. TENSOR RANK.

In dealing with general coordinate transformations between arbitrary curvilinear coordinate systems, the tensors defined are known as *general tensors*. When attention is restricted to transformations from one homogeneous coordinate system to another, the tensors involved are referred to as *Cartesian tensors*. Since much of the theory of continuum mechanics may be developed in terms of Cartesian tensors, the word "tensor" in this book means "Cartesian tensor" unless specifically stated otherwise.

Tensors may be classified by *rank*, or *order*, according to the particular form of the transformation law they obey. This same classification is also reflected in the number of components a given tensor possesses in an n-dimensional space. Thus in a three-dimensional Euclidean space such as ordinary physical space, the number of components of a tensor is 3^N, where N is the order of the tensor. Accordingly a tensor of *order zero* is specified in any coordinate system in three-dimensional space by *one* component. Tensors of order zero are called *scalars*. Physical quantities having magnitude only are represented by scalars. Tensors of *order one* have *three* coordinate components in physical space and are known as *vectors*. Quantities possessing both magnitude and direction are represented by vectors. *Second-order* tensors correspond to *dyadics*. Several important quantities in continuum mechanics are represented by tensors of rank two. Higher order tensors such as *triadics*, or tensors of order three, and *tetradics*, or tensors of order four are also defined and appear often in the mathematics of continuum mechanics.

1.3 VECTORS AND SCALARS

Certain physical quantities, such as force and velocity, which possess both magnitude and direction, may be represented in a three-dimensional space by *directed line segments* that obey the *parallelogram law of addition.* Such directed line segments are the geometrical representations of first-order tensors and are called *vectors.* Pictorially, a vector is simply an arrow pointing in the appropriate direction and having a length proportional to the magnitude of the vector. *Equal vectors* have the same direction and equal magnitudes. A *unit* vector is a vector of unit length. The *null* or *zero* vector is one having zero length and an unspecified direction. The *negative* of a vector is that vector having the same magnitude but opposite direction.

Those physical quantities, such as mass and energy, which possess magnitude only are represented by tensors of order zero which are called *scalars.*

In the *symbolic,* or *Gibbs* notation, vectors are designated by bold-faced letters such as $\mathbf{a}, \mathbf{b}$, etc. Scalars are denoted by italic letters such as a, b, λ, etc. Unit vectors are further distinguished by a caret placed over the bold-faced letter. In Fig. 1-1, arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ are shown along with the unit vector $\hat{\mathbf{e}}$ and the pair of equal vectors $\mathbf{c}$ and $\mathbf{d}$.

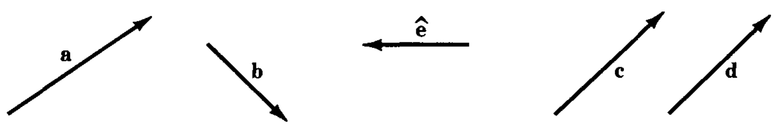


Fig. 1-1

The magnitude of an arbitrary vector $\mathbf{a}$ is written simply as a, or for emphasis it may be denoted by the vector symbol between vertical bars as $|\mathbf{a}|$.

1.4 VECTOR ADDITION. MULTIPLICATION OF A VECTOR BY A SCALAR

Vector addition obeys the *parallelogram law,* which defines the vector sum of two vectors as the diagonal of a parallelogram having the component vectors as adjacent sides. This law for vector addition is equivalent to the *triangle rule* which defines the sum of two vectors as the vector extending from the tail of the first to the head of the second when the summed vectors are adjoined head to tail. The graphical construction for the addition of $\mathbf{a}$ and $\mathbf{b}$ by the parallelogram law is shown in Fig. 1-2(a). Algebraically, the addition process is expressed by the vector equation

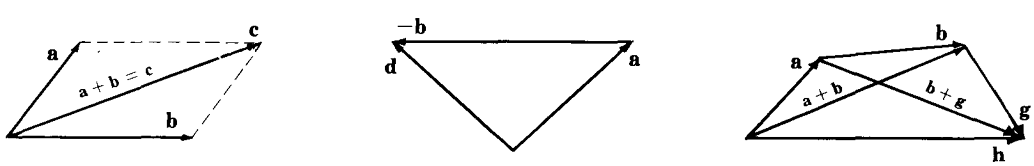
$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a} = \mathbf{c} \tag{1.1}$$

Vector subtraction is accomplished by addition of the negative vector as shown, for example, in Fig. 1-2(b) where the triangle rule is used. Thus

$$\mathbf{a}-\mathbf{b} = -\mathbf{b}+\mathbf{a} = \mathbf{d} \tag{1.2}$$

The operations of vector addition and subtraction are commutative and associative as illustrated in Fig. 1-2(c), for which the appropriate equations are

$$(\mathbf{a}+\mathbf{b})+\mathbf{g} = \mathbf{a}+(\mathbf{b}+\mathbf{g}) = \mathbf{h} \tag{1.3}$$



(a) (b) (c)

Fig. 1-2

Multiplication of a *vector* by a *scalar* produces in general a new vector having the same direction as the original but a different length. Exceptions are multiplication by zero to produce the null vector, and multiplication by unity which does not change a vector. Multiplication of the vector **b** by the scalar m results in one of the three possible cases shown in Fig. 1-3, depending upon the numerical value of m.

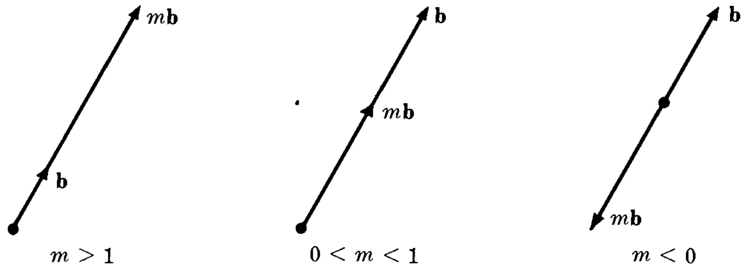


Fig. 1-3

Multiplication of a vector by a scalar is associative and distributive. Thus

$$m(n\mathbf{b}) = (mn)\mathbf{b} = n(m\mathbf{b}) \tag{1.4}$$

$$(m+n)\mathbf{b} = (n+m)\mathbf{b} = m\mathbf{b} + n\mathbf{b} \tag{1.5}$$

$$m(\mathbf{a}+\mathbf{b}) = m(\mathbf{b}+\mathbf{a}) = m\mathbf{a} + m\mathbf{b} \tag{1.6}$$

In the important case of a vector multiplied by the reciprocal of its magnitude, the result is a *unit vector* in the direction of the original vector. This relationship is expressed by the equation

$$\hat{\mathbf{b}} = \mathbf{b}/b \tag{1.7}$$

1.5 DOT AND CROSS PRODUCTS OF VECTORS

The *dot* or *scalar product* of two vectors **a** and **b** is the scalar

$$\lambda = \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = ab\cos\theta \tag{1.8}$$

in which θ is the smaller angle between the two vectors as shown in Fig. 1-4(a). The dot product of **a** with a unit vector $\hat{\mathbf{e}}$ gives the projection of **a** in the direction of $\hat{\mathbf{e}}$.

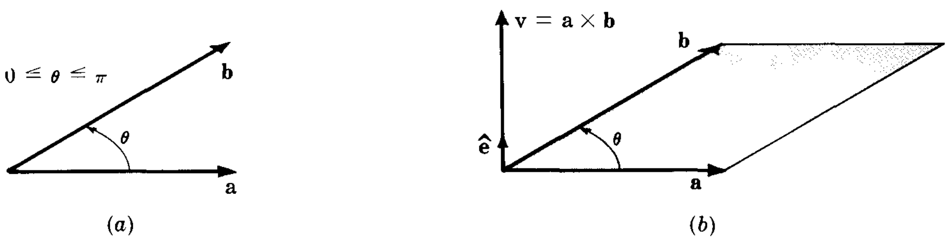


Fig. 1-4

The *cross* or *vector product* of **a** into **b** is the vector **v** given by

$$\mathbf{v} = \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = (ab\sin\theta)\,\hat{\mathbf{e}} \tag{1.9}$$

in which θ is the angle less than 180° between the vectors **a** and **b**, and $\hat{\mathbf{e}}$ is a unit vector perpendicular to their plane such that a right-handed rotation about $\hat{\mathbf{e}}$ through the angle θ carries **a** into **b**. The magnitude of **v** is equal to the area of the parallelogram having **a** and **b** as adjacent sides, shown shaded in Fig. 1-4(b). The cross product is not commutative.

The *scalar triple product* is a dot product of two vectors, one of which is a cross product.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \;=\; (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \;=\; \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \lambda \tag{1.10}$$

As indicated by (*1.10*) the dot and cross operation may be interchanged in this product. Also, since the cross operation must be carried out first, the parentheses are unnecessary and may be deleted as shown. This product is sometimes written [**abc**] and called the *box product*. The magnitude λ of the scalar triple product is equal to the volume of the parallelepiped having **a**, **b**, **c** as coterminous edges.

The *vector triple product* is a cross product of two vectors, one of which is itself a cross product. The following identity is frequently useful in expressing the product of **a** crossed into $\mathbf{b}\times\mathbf{c}$.

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \;=\; (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \;=\; \mathbf{w} \tag{1.11}$$

From (*1.11*), the product vector **w** is observed to lie in the plane of **b** and **c**.

1.6 DYADS AND DYADICS

The *indeterminate vector product* of **a** and **b**, defined by writing the vectors in juxtaposition as **ab** is called a *dyad*. The indeterminate product is not in general commutative, i.e. $\mathbf{ab} \neq \mathbf{ba}$. The first vector in a dyad is known as the *antecedent*, the second is called the *consequent*. A *dyadic* D corresponds to a tensor of order two and may always be represented as a finite sum of dyads

$$\mathsf{D} \;=\; \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_N\mathbf{b}_N \tag{1.12}$$

which is, however, never unique. In symbolic notation, dyadics are denoted by bold-faced sans-serif letters as above.

If in each dyad of (*1.12*) the antecedents and consequents are interchanged, the resulting dyadic is called the *conjugate dyadic* of D and is written

$$\mathsf{D}_c \;=\; \mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 + \cdots + \mathbf{b}_N\mathbf{a}_N \tag{1.13}$$

If each dyad of D in (*1.12*) is replaced by the dot product of the two vectors, the result is a scalar known as the *scalar of the dyadic* D and is written

$$\mathsf{D}_s \;=\; \mathbf{a}_1\cdot\mathbf{b}_1 + \mathbf{a}_2\cdot\mathbf{b}_2 + \cdots + \mathbf{a}_N\cdot\mathbf{b}_N \tag{1.14}$$

If each dyad of D in (*1.12*) is replaced by the cross product of the two vectors, the result is called the *vector of the dyadic* D and is written

$$\mathsf{D}_v \;=\; \mathbf{a}_1\times\mathbf{b}_1 + \mathbf{a}_2\times\mathbf{b}_2 + \cdots + \mathbf{a}_N\times\mathbf{b}_N \tag{1.15}$$

It can be shown that D_c, D_s and D_v are independent of the representation (*1.12*).

The indeterminate vector product obeys the distributive laws

$$\mathbf{a}(\mathbf{b}+\mathbf{c}) \;=\; \mathbf{ab} + \mathbf{ac} \tag{1.16}$$

$$(\mathbf{a}+\mathbf{b})\mathbf{c} \;=\; \mathbf{ac} + \mathbf{bc} \tag{1.17}$$

$$(\mathbf{a}+\mathbf{b})(\mathbf{c}+\mathbf{d}) \;=\; \mathbf{ac} + \mathbf{ad} + \mathbf{bc} + \mathbf{bd} \tag{1.18}$$

and if λ and μ are any scalars,

$$(\lambda+\mu)\mathbf{ab} \;=\; \lambda\mathbf{ab} + \mu\mathbf{ab} \tag{1.19}$$

$$(\lambda\mathbf{a})\mathbf{b} \;=\; \mathbf{a}(\lambda\mathbf{b}) \;=\; \lambda\mathbf{ab} \tag{1.20}$$

If $\mathbf{v}$ is any vector, the dot products $\mathbf{v}\cdot\mathbf{D}$ and $\mathbf{D}\cdot\mathbf{v}$ are the vectors defined respectively by

$$\mathbf{v}\cdot\mathbf{D} = (\mathbf{v}\cdot\mathbf{a}_1)\mathbf{b}_1 + (\mathbf{v}\cdot\mathbf{a}_2)\mathbf{b}_2 + \cdots + (\mathbf{v}\cdot\mathbf{a}_N)\mathbf{b}_N = \mathbf{u} \tag{1.21}$$

$$\mathbf{D}\cdot\mathbf{v} = \mathbf{a}_1(\mathbf{b}_1\cdot\mathbf{v}) + \mathbf{a}_2(\mathbf{b}_2\cdot\mathbf{v}) + \cdots + \mathbf{a}_N(\mathbf{b}_N\cdot\mathbf{v}) = \mathbf{w} \tag{1.22}$$

In (*1.21*) $\mathbf{D}$ is called the *postfactor*, and in (*1.22*) it is called the *prefactor*. Two dyadics $\mathbf{D}$ and $\mathbf{E}$ are *equal* if and only if for every vector $\mathbf{v}$, either

$$\mathbf{v}\cdot\mathbf{D} = \mathbf{v}\cdot\mathbf{E} \quad\text{or}\quad \mathbf{D}\cdot\mathbf{v} = \mathbf{E}\cdot\mathbf{v} \tag{1.23}$$

The *unit dyadic*, or *idemfactor* $\mathbf{I}$, is the dyadic which can be represented as

$$\mathbf{I} = \hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 \tag{1.24}$$

where $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ constitute any orthonormal basis for three-dimensional Euclidean space (see Section 1.7). The dyadic $\mathbf{I}$ is characterized by the property

$$\mathbf{I}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{I} = \mathbf{v} \tag{1.25}$$

for all vectors $\mathbf{v}$.

The cross products $\mathbf{v}\times\mathbf{D}$ and $\mathbf{D}\times\mathbf{v}$ are the dyadics defined respectively by

$$\mathbf{v}\times\mathbf{D} = (\mathbf{v}\times\mathbf{a}_1)\mathbf{b}_1 + (\mathbf{v}\times\mathbf{a}_2)\mathbf{b}_2 + \cdots + (\mathbf{v}\times\mathbf{a}_N)\mathbf{b}_N = \mathbf{F} \tag{1.26}$$

$$\mathbf{D}\times\mathbf{v} = \mathbf{a}_1(\mathbf{b}_1\times\mathbf{v}) + \mathbf{a}_2(\mathbf{b}_2\times\mathbf{v}) + \cdots + \mathbf{a}_N(\mathbf{b}_N\times\mathbf{v}) = \mathbf{G} \tag{1.27}$$

The dot product of the dyads $\mathbf{ab}$ and $\mathbf{cd}$ is the dyad defined by

$$\mathbf{ab}\cdot\mathbf{cd} = (\mathbf{b}\cdot\mathbf{c})\mathbf{ad} \tag{1.28}$$

From (*1.28*), the dot product of any two dyadics $\mathbf{D}$ and $\mathbf{E}$ is the dyadic

$$\begin{aligned}\mathbf{D}\cdot\mathbf{E} &= (\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_N\mathbf{b}_N)\cdot(\mathbf{c}_1\mathbf{d}_1 + \mathbf{c}_2\mathbf{d}_2 + \cdots + \mathbf{c}_N\mathbf{d}_N)\\ &= (\mathbf{b}_1\cdot\mathbf{c}_1)\mathbf{a}_1\mathbf{d}_1 + (\mathbf{b}_1\cdot\mathbf{c}_2)\mathbf{a}_1\mathbf{d}_2 + \cdots + (\mathbf{b}_N\cdot\mathbf{c}_N)\mathbf{a}_N\mathbf{d}_N = \mathbf{G}\end{aligned} \tag{1.29}$$

The dyadics $\mathbf{D}$ and $\mathbf{E}$ are said to be *reciprocal* of each other if

$$\mathbf{E}\cdot\mathbf{D} = \mathbf{D}\cdot\mathbf{E} = \mathbf{I} \tag{1.30}$$

For reciprocal dyadics, the notation $\mathbf{E} = \mathbf{D}^{-1}$ and $\mathbf{D} = \mathbf{E}^{-1}$ is often used.

Double dot and cross products are also defined for the dyads $\mathbf{ab}$ and $\mathbf{cd}$ as follows,

$$\mathbf{ab} : \mathbf{cd} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) = \lambda, \quad\text{a scalar} \tag{1.31}$$

$$\mathbf{ab} \overset{\times}{\cdot} \mathbf{cd} = (\mathbf{a}\times\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) = \mathbf{h}, \quad\text{a vector} \tag{1.32}$$

$$\mathbf{ab} \underset{\times}{\cdot} \mathbf{cd} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\times\mathbf{d}) = \mathbf{g}, \quad\text{a vector} \tag{1.33}$$

$$\mathbf{ab} \overset{\times}{\times} \mathbf{cd} = (\mathbf{a}\times\mathbf{c})(\mathbf{b}\times\mathbf{d}) = \mathbf{uw}, \quad\text{a dyad} \tag{1.34}$$

From these definitions, double dot and cross products of dyadics may be readily developed. Also, some authors use the double dot product defined by

$$\mathbf{ab}\cdot\cdot\,\mathbf{cd} = (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) = \lambda, \quad\text{a scalar} \tag{1.35}$$

A dyadic $\mathbf{D}$ is said to be *self-conjugate*, or *symmetric*, if

$$\mathbf{D} = \mathbf{D}_c \tag{1.36}$$

and *anti-self-conjugate*, or *anti-symmetric*, if

$$\mathbf{D} = -\mathbf{D}_c \tag{1.37}$$

Every dyadic may be expressed uniquely as the sum of a symmetric and anti-symmetric dyadic. For the arbitrary dyadic $\mathbf{D}$ the decomposition is

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{D} + \mathbf{D}_c) + \tfrac{1}{2}(\mathbf{D} - \mathbf{D}_c) = \mathbf{G} + \mathbf{H} \tag{1.38}$$

for which $$\mathbf{G}_c = \tfrac{1}{2}(\mathbf{D}_c + (\mathbf{D}_c)_c) = \tfrac{1}{2}(\mathbf{D}_c + \mathbf{D}) = \mathbf{G} \quad \text{(symmetric)} \tag{1.39}$$

and $$\mathbf{H}_c = \tfrac{1}{2}(\mathbf{D}_c - (\mathbf{D}_c)_c) = \tfrac{1}{2}(\mathbf{D}_c - \mathbf{D}) = -\mathbf{H} \quad \text{(anti-symmetric)} \tag{1.40}$$

Uniqueness is established by assuming a second decomposition, $\mathbf{D} = \mathbf{G}^* + \mathbf{H}^*$. Then

$$\mathbf{G}^* + \mathbf{H}^* = \mathbf{G} + \mathbf{H} \tag{1.41}$$

and the conjugate of this equation is

$$\mathbf{G}^* - \mathbf{H}^* = \mathbf{G} - \mathbf{H} \tag{1.42}$$

Adding and subtracting (*1.41*) and (*1.42*) in turn yields respectively the desired equalities, $\mathbf{G}^* = \mathbf{G}$ and $\mathbf{H}^* = \mathbf{H}$.

1.7 COORDINATE SYSTEMS. BASE VECTORS. UNIT VECTOR TRIADS

A vector may be defined with respect to a particular coordinate system by specifying the *components* of the vector in that system. The choice of coordinate system is arbitrary, but in certain situations a particular choice may be advantageous. The reference system of coordinate axes provides units for measuring vector magnitudes and assigns directions in space by which the orientation of vectors may be determined.

The well-known *rectangular Cartesian coordinate system* is often represented by the mutually perpendicular axes, *Oxyz* shown in Fig. 1-5. Any vector **v** in this system may be expressed as a linear combination of three arbitrary, nonzero, noncoplanar vectors of the system, which are called *base vectors*. For base vectors **a**, **b**, **c** and suitably chosen scalar coefficients λ, μ, ν the vector **v** is given by

$$\mathbf{v} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} \tag{1.43}$$

Base vectors are by hypothesis linearly independent, i.e. the equation

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = 0 \tag{1.44}$$

is satisfied only if $\lambda = \mu = \nu = 0$. A set of base vectors in a given coordinate system is said to constitute a *basis* for that system.

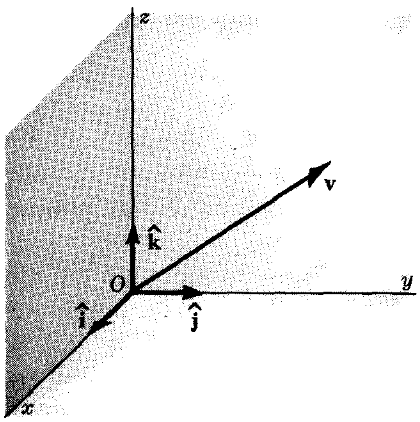


Fig. 1-5

The most frequent choice of base vectors for the rectangular Cartesian system is the set of unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ along the coordinate axes as shown in Fig. 1-5. These base vectors constitute a right-handed *unit vector triad*, for which

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \tag{1.45}$$

and
$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$$
$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0 \tag{1.46}$$

Such a set of base vectors is often called an *orthonormal basis*.

In terms of the unit triad $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, the vector **v** shown in Fig. 1-6 below may be expressed by

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \tag{1.47}$$

in which the Cartesian components

$$v_x = \mathbf{v}\cdot\hat{\mathbf{i}} = v\cos\alpha$$
$$v_y = \mathbf{v}\cdot\hat{\mathbf{j}} = v\cos\beta$$
$$v_z = \mathbf{v}\cdot\hat{\mathbf{k}} = v\cos\gamma$$

are the projections of $\mathbf{v}$ onto the coordinate axes. The unit vector in the direction of $\mathbf{v}$ is given according to (*1.7*) by

$$\hat{\mathbf{e}}_v = \mathbf{v}/v = (\cos\alpha)\hat{\mathbf{i}} + (\cos\beta)\hat{\mathbf{j}} + (\cos\gamma)\hat{\mathbf{k}} \qquad (1.48)$$

Since $\mathbf{v}$ is arbitrary, it follows that any unit vector will have the *direction cosines* of that vector as its *Cartesian components.*

In Cartesian component form the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a}\cdot\mathbf{b} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}})\cdot(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) = a_xb_x + a_yb_y + a_zb_z \qquad (1.49)$$

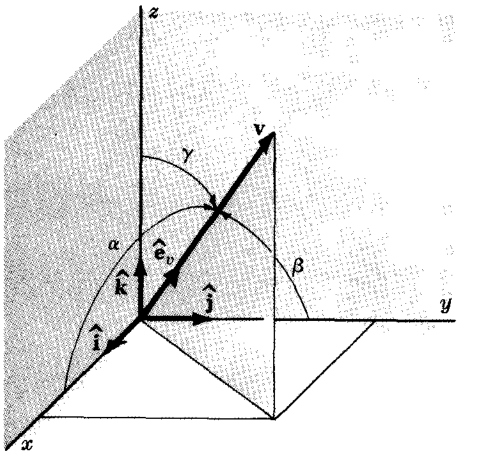


Fig. 1-6

For the same two vectors, the cross product $\mathbf{a}\times\mathbf{b}$ is

$$\mathbf{a}\times\mathbf{b} = (a_yb_z - a_zb_y)\hat{\mathbf{i}} + (a_zb_x - a_xb_z)\hat{\mathbf{j}} + (a_xb_y - a_yb_x)\hat{\mathbf{k}} \qquad (1.50)$$

This result is often presented in the determinant form

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (1.51)$$

in which the elements are treated as ordinary numbers. The triple scalar product may also be represented in component form by the determinant

$$[\mathbf{abc}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \qquad (1.52)$$

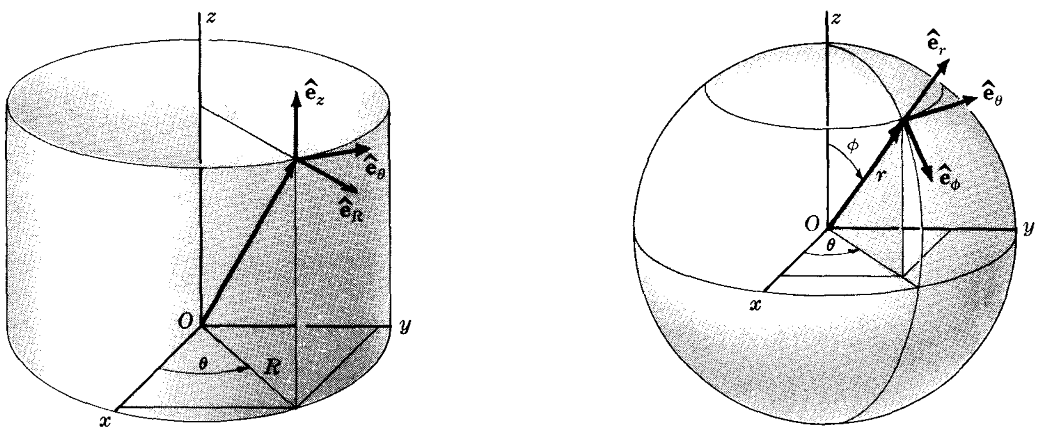
In Cartesian component form, the dyad $\mathbf{ab}$ is given by

$$\begin{aligned}\mathbf{ab} &= (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}})(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \\ &= a_xb_x\hat{\mathbf{i}}\hat{\mathbf{i}} + a_xb_y\hat{\mathbf{i}}\hat{\mathbf{j}} + a_xb_z\hat{\mathbf{i}}\hat{\mathbf{k}} \\ &\quad + a_yb_x\hat{\mathbf{j}}\hat{\mathbf{i}} + a_yb_y\hat{\mathbf{j}}\hat{\mathbf{j}} + a_yb_z\hat{\mathbf{j}}\hat{\mathbf{k}} \\ &\quad + a_zb_x\hat{\mathbf{k}}\hat{\mathbf{i}} + a_zb_y\hat{\mathbf{k}}\hat{\mathbf{j}} + a_zb_z\hat{\mathbf{k}}\hat{\mathbf{k}}\end{aligned} \qquad (1.53)$$

Because of the *nine* terms involved, (*1.53*) is known as the *nonion form* of the dyad $\mathbf{ab}$. It is possible to put any dyadic into nonion form. The nonion form of the idemfactor in terms of the unit triad $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ is given by

$$\mathbf{I} = \hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}} \qquad (1.54)$$

In addition to the rectangular Cartesian coordinate system already discussed, curvilinear coordinate systems such as the cylindrical (R, θ, z) and spherical (r, θ, ϕ) systems shown in Fig. 1-7 below are also widely used. Unit triads $(\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$ and $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi)$ of base vectors illustrated in the figure are associated with these systems. However, the base vectors here do not all have fixed directions and are therefore, in general, functions of position.



(a) Cylindrical (b) Spherical

Fig. 1-7

1.8 LINEAR VECTOR FUNCTIONS. DYADICS AS LINEAR VECTOR OPERATORS

A vector **a** is said to be a function of a second vector **b** if **a** is determined whenever **b** is given. This functional relationship is expressed by the equation

$$\mathbf{a} = \mathbf{f}(\mathbf{b}) \tag{1.55}$$

The function **f** is said to be linear when the conditions

$$\mathbf{f}(\mathbf{b}+\mathbf{c}) = \mathbf{f}(\mathbf{b}) + \mathbf{f}(\mathbf{c}) \tag{1.56}$$

$$\mathbf{f}(\lambda\mathbf{b}) = \lambda\mathbf{f}(\mathbf{b}) \tag{1.57}$$

are satisfied for all vectors **b** and **c**, and for any scalar λ.

Writing **b** in Cartesian component form, equation (*1.55*) becomes

$$\mathbf{a} = \mathbf{f}(b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}) \tag{1.58}$$

which, if **f** is linear, may be written

$$\mathbf{a} = b_x\mathbf{f}(\hat{\mathbf{i}}) + b_y\mathbf{f}(\hat{\mathbf{j}}) + b_z\mathbf{f}(\hat{\mathbf{k}}) \tag{1.59}$$

In (*1.59*) let $\mathbf{f}(\hat{\mathbf{i}}) = \mathbf{u}$, $\mathbf{f}(\hat{\mathbf{j}}) = \mathbf{v}$, $\mathbf{f}(\hat{\mathbf{k}}) = \mathbf{w}$, so that now

$$\mathbf{a} = \mathbf{u}(\hat{\mathbf{i}}\cdot\mathbf{b}) + \mathbf{v}(\hat{\mathbf{j}}\cdot\mathbf{b}) + \mathbf{w}(\hat{\mathbf{k}}\cdot\mathbf{b}) = (\mathbf{u}\hat{\mathbf{i}} + \mathbf{v}\hat{\mathbf{j}} + \mathbf{w}\hat{\mathbf{k}})\cdot\mathbf{b} \tag{1.60}$$

which is recognized as a dyadic-vector dot product and may be written

$$\mathbf{a} = \mathsf{D}\cdot\mathbf{b} \tag{1.61}$$

where $\mathsf{D} = \mathbf{u}\hat{\mathbf{i}} + \mathbf{v}\hat{\mathbf{j}} + \mathbf{w}\hat{\mathbf{k}}$. This demonstrates that any linear vector function **f** may be expressed as a dyadic-vector product. In (*1.61*) the dyadic D serves as a *linear vector operator* which operates on the *argument* vector **b** to produce the *image* vector **a**.

1.9 INDICIAL NOTATION. RANGE AND SUMMATION CONVENTIONS

The components of a tensor of any order, and indeed the tensor itself, may be represented clearly and concisely by the use of the *indicial notation*. In this notation, letter indices, either subscripts or superscripts, are appended to the *generic* or *kernel* letter representing the tensor quantity of interest. Typical examples illustrating use of indices are the tensor symbols

$$a_i, \; b^j, \; T_{ij}, \; F_i{}^j, \; \epsilon_{ijk}, \; R^{pq}$$

In the "mixed" form, where both subscripts and superscripts appear, the dot shows that j is the second index.

Under the rules of indicial notation, a letter index may occur either *once* or *twice* in a given term. When an index occurs unrepeated in a term, that index is understood to take on the values $1, 2, \ldots, N$ where N is a specified integer that determines the *range* of the index. Unrepeated indices are known as *free* indices. The tensorial rank of a given term is equal to the number of free indices appearing in that term. Also, correctly written tensor equations have the same letters as free indices in every term.

When an index appears *twice* in a term, that index is understood to take on all the values of its range, and the resulting terms *summed.* In this so-called *summation convention,* repeated indices are often referred to as *dummy indices,* since their replacement by any other letter not appearing as a free index does not change the meaning of the term in which they occur. In general, no index occurs more than twice in a properly written term. If it is absolutely necessary to use some index more than twice to satisfactorily express a certain quantity, the summation convention must be suspended.

The number and location of the free indices reveal directly the exact tensorial character of the quantity expressed in the indicial notation. Tensors of *first order* are denoted by kernel letters bearing *one free index.* Thus the arbitrary vector **a** is represented by a symbol having a single subscript or superscript, i.e. in one or the other of the two forms,

$$a_i, \quad a^i$$

The following terms, having only one free index, are also recognized as first-order tensor quantities:

$$a_{ij}b_j, \quad F_{ikk}, \quad R^{p}_{\cdot qp}, \quad \epsilon_{ijk}u_jv_k$$

Second-order tensors are denoted by symbols having *two* free indices. Thus the arbitrary dyadic **D** will appear in one of the three possible forms

$$D^{ij}, \; D_i^{\cdot j} \quad \text{or} \quad D^i_{\cdot j}, \; D_{ij}$$

In the "mixed" form, the dot shows that j is the second index. Second-order tensor quantities may also appear in various forms as, for example,

$$A_{ijip}, \quad B^{ij}_{\cdot\cdot jk}, \quad \delta_{ij}u_kv_k$$

By a logical continuation of the above scheme, *third-order* tensors are expressed by symbols with *three* free indices. Also, a symbol such as λ which has no indices attached, represents a scalar, or tensor of zero order.

In ordinary physical space a *basis* is composed of three, noncoplanar vectors, and so any vector in this space is completely specified by its three components. Therefore the range on the index of a_i, which represents a vector in physical three-space, is $1, 2, 3$. Accordingly the symbol a_i is understood to represent the three components a_1, a_2, a_3. Also, a_i is sometimes interpreted to represent the ith component of the vector or indeed to represent the vector itself. For a range of three on both indices, the symbol A_{ij} represents nine components (of the second-order tensor (dyadic) **A**). The tensor A_{ij} is often presented explicitly by giving the nine components in a square array enclosed by large parentheses as

$$A_{ij} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \qquad (1.62)$$

In the same way, the components of a first-order tensor (vector) in three-space may be displayed explicitly by a row or column arrangement of the form

$$a_i = (a_1, a_2, a_3) \qquad \text{or} \qquad a_i = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{1.63}$$

In general, for a range of N, an nth order tensor will have N^n components.

The usefulness of the indicial notation in presenting systems of equations in compact form is illustrated by the following two typical examples. For a range of three on both i and j the indicial equation

$$x_i = c_{ij} z_j \tag{1.64}$$

represents in expanded form the three equations

$$\begin{aligned} x_1 &= c_{11}z_1 + c_{12}z_2 + c_{13}z_3 \\ x_2 &= c_{21}z_1 + c_{22}z_2 + c_{23}z_3 \\ x_3 &= c_{31}z_1 + c_{32}z_2 + c_{33}z_3 \end{aligned} \tag{1.65}$$

For a range of two on i and j, the indicial equation

$$A_{ij} = B_{ip} C_{jq} D_{pq} \tag{1.66}$$

represents, in expanded form, the four equations

$$\begin{aligned} A_{11} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\ A_{12} &= B_{11}C_{21}D_{11} + B_{11}C_{22}D_{12} + B_{12}C_{21}D_{21} + B_{12}C_{22}D_{22} \\ A_{21} &= B_{21}C_{11}D_{11} + B_{21}C_{12}D_{12} + B_{22}C_{11}D_{21} + B_{22}C_{12}D_{22} \\ A_{22} &= B_{21}C_{21}D_{11} + B_{21}C_{22}D_{12} + B_{22}C_{21}D_{21} + B_{22}C_{22}D_{22} \end{aligned} \tag{1.67}$$

For a range of three on both i and j, (*1.66*) would represent nine equations, each having nine terms on the right-hand side.

1.10 SUMMATION CONVENTION USED WITH SYMBOLIC NOTATION

The summation convention is very often employed in connection with the representation of vectors and tensors by *indexed base vectors* written in the symbolic notation. Thus if the rectangular Cartesian axes and unit base vectors of Fig. 1-5 are relabeled as shown by Fig. 1-8, the arbitrary vector $\mathbf{v}$ may be written

$$\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3 \tag{1.68}$$

in which v_1, v_2, v_3 are the rectangular Cartesian components of $\mathbf{v}$. Applying the summation convention to (*1.68*), the equation may be written in the abbreviated form

$$\mathbf{v} = v_i\hat{\mathbf{e}}_i \tag{1.69}$$

where i is a summed index. The notation here is essentially *symbolic*, but with the added feature of the *summation convention*. In such a "combination" style of notation, tensor character is not given by the *free indices rule* as it is in true indicial notation.

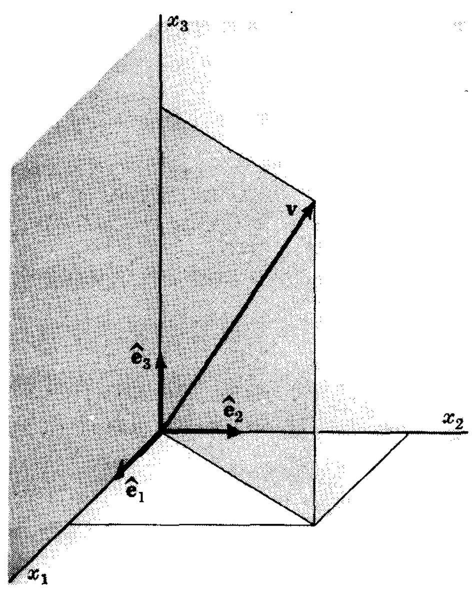


Fig. 1-8

Second-order tensors may also be represented by summation on indexed base vectors. Accordingly the dyad **ab** given in nonion form by (*1.53*) may be written

$$\mathbf{ab} = (a_i\hat{\mathbf{e}}_i)(b_j\hat{\mathbf{e}}_j) = a_ib_j\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \tag{1.70}$$

It is essential that the sequence of the base vectors be preserved in this expression. In similar fashion, the nonion form of the arbitrary dyadic **D** may be expressed in compact notation by

$$\mathbf{D} = D_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \tag{1.71}$$

1.11 COORDINATE TRANSFORMATIONS. GENERAL TENSORS

Let x^i represent the arbitrary system of coordinates x^1, x^2, x^3 in a three-dimensional Euclidean space, and let θ^i represent any other coordinate system $\theta^1, \theta^2, \theta^3$ in the same space. Here the numerical superscripts are labels and not exponents. Powers of x may be expressed by use of parentheses as in $(x)^2$ or $(x)^3$. The letter superscripts are indices as already noted. The *coordinate transformation equations*

$$\theta^i = \theta^i(x^1, x^2, x^3) \tag{1.72}$$

assign to any point (x^1, x^2, x^3) in the x^i system a new set of coordinates $(\theta^1, \theta^2, \theta^3)$ in the θ^i system. The functions θ^i relating the two sets of variables (coordinates) are assumed to be single-valued, continuous, differentiable functions. The determinant

$$J = \begin{vmatrix} \dfrac{\partial\theta^1}{\partial x^1} & \dfrac{\partial\theta^1}{\partial x^2} & \dfrac{\partial\theta^1}{\partial x^3} \\[2ex] \dfrac{\partial\theta^2}{\partial x^1} & \dfrac{\partial\theta^2}{\partial x^2} & \dfrac{\partial\theta^2}{\partial x^3} \\[2ex] \dfrac{\partial\theta^3}{\partial x^1} & \dfrac{\partial\theta^3}{\partial x^2} & \dfrac{\partial\theta^3}{\partial x^3} \end{vmatrix} \tag{1.73}$$

or, in compact form,

$$J = \left|\frac{\partial\theta^i}{\partial x^j}\right| \tag{1.74}$$

is called the *Jacobian* of the transformation. If the Jacobian does not vanish, (*1.72*) possesses a unique inverse set of the form

$$x^i = x^i(\theta^1, \theta^2, \theta^3) \tag{1.75}$$

The coordinate systems represented by x^i and θ^i in (*1.72*) and (*1.75*) are completely general and may be any curvilinear or Cartesian systems.

From (*1.72*), the differential vector $d\theta^i$ is given by

$$d\theta^i = \frac{\partial\theta^i}{\partial x^j}\,dx^j \tag{1.76}$$

This equation is a prototype of the equation which defines the class of tensors known as *contravariant vectors*. In general, a set of quantities b^i associated with a point P are said to be the components of a *contravariant tensor of order one* if they transform, under a coordinate transformation, according to the equation

$$b'^i = \frac{\partial\theta^i}{\partial x^j}\,b^j \tag{1.77}$$

where the partial derivatives are evaluated at P. In (*1.77*), b^j are the components of the tensor in the x^j coordinate system, while b'^i are the components in the θ^i system. In general tensor theory, contravariant tensors are recognized by the use of superscripts as indices. It is for this reason that the coordinates are labeled x^i here rather than x_i, but it must be noted that it is only the differentials dx^i, and not the coordinates themselves, which have tensor character.

By a logical extension of the tensor concept expressed in (*1.77*), the definition of *contravariant tensors of order two* requires the tensor components to obey the transformation law

$$B'^{ij} = \frac{\partial \theta^i}{\partial x^r}\frac{\partial \theta^j}{\partial x^s} B^{rs} \tag{1.78}$$

Contravariant tensors of third, fourth and higher orders are defined in a similar manner.

The word *contravariant* is used above to distinguish that class of tensors from the class known as *covariant* tensors. In general tensor theory, covariant tensors are recognized by the use of subscripts as indices. The prototype of the *covariant vector* is the partial derivative of a scalar function of the coordinates. Thus if $\phi = \phi(x^1, x^2, x^3)$ is such a function,

$$\frac{\partial \phi}{\partial \theta^i} = \frac{\partial \phi}{\partial x^j}\frac{\partial x^j}{\partial \theta^i} \tag{1.79}$$

In general, a set of quantities b_i are said to be the components of a *covariant tensor of order one* if they transform according to the equation

$$b'_i = \frac{\partial x^j}{\partial \theta^i} b_j \tag{1.80}$$

In (*1.80*), b'_i are the covariant components in the θ^i system, b_i the components in the x_i system. *Second-order covariant tensors* obey the transformation law

$$B'_{ij} = \frac{\partial x^r}{\partial \theta^i}\frac{\partial x^s}{\partial \theta^j} B_{rs} \tag{1.81}$$

Covariant tensors of higher order and *mixed tensors*, such as

$$T'^{r}_{\cdot sp} = \frac{\partial \theta^r}{\partial x^m}\frac{\partial x^n}{\partial \theta^s}\frac{\partial x^q}{\partial \theta^p} T^{m}_{\cdot nq} \tag{1.82}$$

are defined in the obvious way.

1.12 THE METRIC TENSOR. CARTESIAN TENSORS

Let x^i represent a system of rectangular Cartesian coordinates in a Euclidean three-space, and let θ^i represent any system of rectangular or curvilinear coordinates (e.g. cylindrical or spherical coordinates) in the same space. The vector $\mathbf{x}$ having Cartesian components x^i is called the *position vector* of the arbitrary point $P(x^1, x^2, x^3)$ referred to the rectangular Cartesian axes. The square of the differential element of distance between neighboring points $P(\mathbf{x})$ and $Q(\mathbf{x} + d\mathbf{x})$ is given by

$$(ds)^2 = dx^i\, dx^i \tag{1.83}$$

From the coordinate transformation

$$x^i = x^i(\theta^1, \theta^2, \theta^3) \tag{1.84}$$

relating the systems, the distance differential is

$$dx^i = \frac{\partial x^i}{\partial \theta^p} d\theta^p \tag{1.85}$$

and therefore (*1.83*) becomes

$$(ds)^2 = \frac{\partial x^i}{\partial \theta^p}\frac{\partial x^i}{\partial \theta^q}\,d\theta^p\,d\theta^q = g_{pq}\,d\theta^p\,d\theta^q \tag{1.86}$$

where the second-order tensor $g_{pq} = (\partial x^i/\partial \theta^p)(\partial x^i/\partial \theta^q)$ is called the *metric tensor*, or *fundamental tensor* of the space. If θ^i represents a rectangular Cartesian system, say the x'^i system, then

$$g_{pq} = \frac{\partial x^i}{\partial x'^p}\frac{\partial x^i}{\partial x'^q} = \delta_{pq} \tag{1.87}$$

where δ_{pq} is the *Kronecker delta* (see Section 1.13) defined by $\delta_{pq} = 0$ if $p \neq q$ and $\delta_{pq} = 1$ if $p = q$.

Any system of coordinates for which the squared differential element of distance takes the form of (*1.83*) is called a system of *homogeneous coordinates.* Coordinate transformations between systems of homogeneous coordinates are *orthogonal transformations,* and when attention is restricted to such transformations, the tensors so defined are called *Cartesian tensors.* In particular, this is the case for transformation laws between systems of rectangular Cartesian coordinates with a common origin. For Cartesian tensors there is no distinction between contravariant and covariant components and therefore it is customary to use subscripts exclusively in expressions representing Cartesian tensors. As will be shown next, in the transformation laws defining Cartesian tensors, the partial derivatives appearing in general tensor definitions, such as (*1.80*) and (*1.81*), are replaced by constants.

1.13. TRANSFORMATION LAWS FOR CARTESIAN TENSORS. THE KRONECKER DELTA. ORTHOGONALITY CONDITIONS

Let the axes $Ox_1x_2x_3$ and $Ox'_1x'_2x'_3$ represent two rectangular Cartesian coordinate systems with a common origin at an arbitrary point O as shown in Fig. 1-9. The primed system may be imagined to be obtained from the unprimed by a rotation of the axes about the origin, or by a reflection of axes in one of the coordinate planes, or by a combination of these. If the symbol a_{ij} denotes the cosine of the angle between the ith primed and jth unprimed coordinate axes, i.e. $a_{ij} = \cos(x'_i, x_j)$, the relative orientation of the individual axes of each system with respect to the other is conveniently given by the table

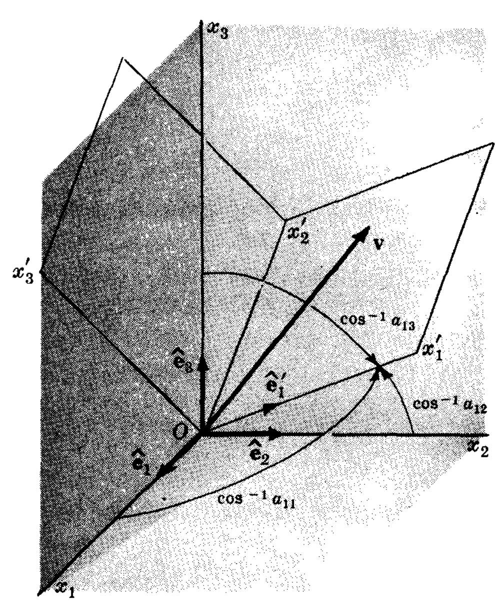


Fig. 1-9

	x_1	x_2	x_3
x'_1	a_{11}	a_{12}	a_{13}
x'_2	a_{21}	a_{22}	a_{23}
x'_3	a_{31}	a_{32}	a_{33}

or alternatively by the transformation tensor

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

From this definition of a_{ij}, the unit vector $\hat{\mathbf{e}}'_1$ along the x'_1 axis is given according to (*1.48*) and the summation convention by

$$\hat{\mathbf{e}}'_1 = a_{11}\hat{\mathbf{e}}_1 + a_{12}\hat{\mathbf{e}}_2 + a_{13}\hat{\mathbf{e}}_3 = a_{1j}\hat{\mathbf{e}}_j \tag{1.88}$$

An obvious generalization of this equation gives the arbitrary unit base vector $\hat{\mathbf{e}}'_i$ as

$$\hat{\mathbf{e}}'_i = a_{ij}\hat{\mathbf{e}}_j \tag{1.89}$$

In component form, the arbitrary vector $\mathbf{v}$ shown in Fig. 1-9 may be expressed in the unprimed system by the equation

$$\mathbf{v} = v_j\hat{\mathbf{e}}_j \tag{1.90}$$

and in the primed system by

$$\mathbf{v} = v'_i\hat{\mathbf{e}}'_i \tag{1.91}$$

Replacing $\hat{\mathbf{e}}'_i$ in (*1.91*) by its equivalent form (*1.89*) yields the result

$$\mathbf{v} = v'_i a_{ij}\hat{\mathbf{e}}_j \tag{1.92}$$

Comparing (*1.92*) with (*1.90*) reveals that the vector components in the primed and unprimed systems are related by the equations

$$v_j = a_{ij}v'_i \tag{1.93}$$

The expression (*1.93*) is the *transformation law* for first-order Cartesian tensors, and as such is seen to be a special case of the general form of first-order tensor transformations, expressed by (*1.80*) and (*1.77*). By interchanging the roles of the primed and unprimed base vectors in the above development, the inverse of (*1.93*) is found to be

$$v'_i = a_{ij}v_j \tag{1.94}$$

It is important to note that in (*1.93*) the free index on a_{ij} appears as the second index. In (*1.94*), however, the free index appears as the first index.

By an appropriate choice of dummy indices, (*1.93*) and (*1.94*) may be combined to produce the equation

$$v_j = a_{ij}a_{ik}v_k \tag{1.95}$$

Since the vector $\mathbf{v}$ is arbitrary, (*1.95*) must reduce to the identity $v_j = v_j$. Therefore the coefficient $a_{ij}a_{ik}$, whose value depends upon the subscripts j and k, must equal 1 or 0 according to whether the numerical values of j and k are the same or different. The *Kronecker delta*, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{1.96}$$

may be used to represent quantities such as $a_{ij}a_{ik}$. Thus with the help of the Kronecker delta the conditions on the coefficient in (*1.95*) may be written

$$a_{ij}a_{ik} = \delta_{jk} \tag{1.97}$$

In expanded form, (*1.97*) consists of nine equations which are known as the *orthogonality* or *orthonormality conditions* on the direction cosines a_{ij}. Finally, (*1.93*) and (*1.94*) may also be combined to produce $v_i = a_{ij}a_{kj}v'_k$ from which the orthogonality conditions appear in the alternative form

$$a_{ij}a_{kj} = \delta_{ik} \tag{1.98}$$

A linear transformation such as (*1.93*) or (*1.94*), whose coefficients satisfy (*1.97*) or (*1.98*), is said to be an *orthogonal transformation*. Coordinate axes rotations and reflections of the axes in a coordinate plane both lead to orthogonal transformations.

The Kronecker delta is sometimes called the *substitution operator,* since, for example,

$$\delta_{ij}b_j = \delta_{i1}b_1 + \delta_{i2}b_2 + \delta_{i3}b_3 = b_i \tag{1.99}$$

and, likewise,

$$\delta_{ij}F_{ik} = \delta_{1j}F_{1k} + \delta_{2j}F_{2k} + \delta_{3j}F_{3k} = F_{jk} \tag{1.100}$$

It is clear from this property that the Kronecker delta is the indicial counterpart to the symbolic idemfactor **I**, which is given by (*1.54*).

According to the transformation law (*1.94*), the dyad u_iv_j has components in the primed coordinate system given by

$$u'_i v'_j = (a_{ip}u_p)(a_{jq}v_q) = a_{ip}a_{jq}u_pv_q \tag{1.101}$$

In an obvious generalization of (*1.101*), any second-order Cartesian tensor T_{ij} obeys the transformation law

$$T'_{ij} = a_{ip}a_{jq}T_{pq} \tag{1.102}$$

With the help of the orthogonality conditions it is a simple calculation to invert (*1.102*), thereby giving the transformation rule from primed components to unprimed components:

$$T_{ij} = a_{pi}a_{qj}T'_{pq} \tag{1.103}$$

The transformation laws for first and second-order Cartesian tensors generalize for an Nth order Cartesian tensor to

$$T'_{ijk\ldots} = a_{ip}a_{jq}a_{km}\ldots T_{pqm\ldots} \tag{1.104}$$

1.14 ADDITION OF CARTESIAN TENSORS. MULTIPLICATION BY A SCALAR

Cartesian tensors of the same order may be added (or subtracted) component by component in accordance with the rule

$$A_{ijk\ldots} \pm B_{ijk\ldots} = T_{ijk\ldots} \tag{1.105}$$

The sum is a tensor of the same order as those added. Note that like indices appear in the same sequence in each term.

Multiplication of every component of a tensor by a given scalar produces a new tensor of the same order. For the scalar multiplier λ, typical examples written in both indicial and symbolic notation are

$$b_i = \lambda a_i \quad \text{or} \quad \mathbf{b} = \lambda\mathbf{a} \tag{1.106}$$

$$B_{ij} = \lambda A_{ij} \quad \text{or} \quad \mathbf{B} = \lambda\mathbf{A} \tag{1.107}$$

1.15 TENSOR MULTIPLICATION

The *outer product* of two tensors of arbitrary order is the tensor whose components are formed by multiplying each component of one of the tensors by every component of the other. This process produces a tensor having an order which is the sum of the orders of the factor tensors. Typical examples of outer products are

(*a*) $a_ib_j = T_{ij}$ (*c*) $D_{ij}T_{km} = \Phi_{ijkm}$

(*b*) $v_iF_{jk} = \alpha_{ijk}$ (*d*) $\epsilon_{ijk}v_m = \Theta_{ijkm}$

As indicated by the above examples, outer products are formed by simply setting down the factor tensors in juxtaposition. (Note that a dyad is formed from two vectors by this very procedure.)

Contraction of a tensor with respect to two free indices is the operation of assigning to both indices the same letter subscript, thereby changing these indices to dummy indices. Contraction produces a tensor having an order two less than the original. Typical examples of contraction are the following.

(*a*) Contractions of T_{ij} and $u_i v_j$

$$T_{ii} = T_{11} + T_{22} + T_{33}$$

$$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$$

(*b*) Contractions of $E_{ij} a_k$

$$E_{ij} a_j = b_i$$

$$E_{ij} a_i = c_j$$

$$E_{ii} a_k = d_k$$

(*c*) Contractions of $E_{ij} F_{km}$

$$E_{ij} F_{im} = G_{jm} \qquad E_{ij} F_{kk} = P_{ij}$$

$$E_{ij} F_{ki} = H_{jk} \qquad E_{ij} F_{jm} = Q_{im}$$

$$E_{ii} F_{km} = K_{km} \qquad E_{ij} F_{kj} = R_{ik}$$

An *inner product* of two tensors is the result of a contraction, involving one index from each tensor, performed on the outer product of the two tensors. Several inner products important to continuum mechanics are listed here for reference, in both the indicial and symbolic notations.

Outer Product	**Inner Product**	
	Indicial Notation	Symbolic Notation
1. $a_i b_j$	$a_i b_i$	$\mathbf{a} \cdot \mathbf{b}$
2. $a_i E_{jk}$	$a_i E_{ik} = f_k$	$\mathbf{a} \cdot \mathbf{E} = \mathbf{f}$
	$a_i E_{ji} = h_j$	$\mathbf{E} \cdot \mathbf{a} = \mathbf{h}$
3. $E_{ij} F_{km}$	$E_{ij} F_{jm} = G_{im}$	$\mathbf{E} \cdot \mathbf{F} = \mathbf{G}$
4. $E_{ij} E_{km}$	$E_{ij} E_{jm} = B_{im}$	$\mathbf{E} \cdot \mathbf{E} = (\mathbf{E})^2$

Multiple contractions of fourth-order and higher tensors are sometimes useful. Two such examples are

1. $E_{ij} F_{km}$ contracted to $E_{ij} F_{ij}$, or $\mathbf{E} : \mathbf{F}$
2. $E_{ij} E_{km} E_{pq}$ contracted to $E_{ij} E_{jm} E_{mq}$, or $(\mathbf{E})^3$

1.16 VECTOR CROSS PRODUCT. PERMUTATION SYMBOL. DUAL VECTORS

In order to express the cross product $\mathbf{a} \times \mathbf{b}$ in the indicial notation, the third-order tensor ϵ_{ijk}, known as the *permutation symbol* or *alternating tensor*, must be introduced. This useful tensor is defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if the values of } i,j,k \text{ are an } \textit{even permutation} \text{ of } 1,2,3 \text{ (i.e. if they appear in sequence as in the arrangement } 1\,2\,3\,1\,2). \\ -1 & \text{if the values of } i,j,k \text{ are an } \textit{odd permutation} \text{ of } 1,2,3 \text{ (i.e. if they appear in sequence as in the arrangement } 3\,2\,1\,3\,2). \\ 0 & \text{if the values of } i,j,k \text{ are } \textit{not} \text{ a permutation of } 1,2,3 \text{ (i.e. if two or more of the indices have the same value).} \end{cases}$$

From this definition, the cross product $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ is written in indicial notation by

$$\epsilon_{ijk} a_j b_k = c_i \tag{1.108}$$

Using this relationship, the box product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \lambda$ may be written

$$\lambda = \epsilon_{ijk} a_i b_j c_k \tag{1.109}$$

Since the same box product is given in the form of a determinant by *(1.52)*, it is not surprising that the permutation symbol is frequently used to express the value of a 3×3 determinant.

It is worthwhile to note that ϵ_{ijk} obeys the tensor transformation law for third order Cartesian tensors as long as the transformation is a *proper* one ($\det a_{ij} = 1$) such as arises from a rotation of axes. If the transformation is *improper* ($\det a_{ij} = -1$), e.g. a reflection in one of the coordinate planes whereby a right-handed coordinate system is transformed into a left-handed one, a minus sign must be inserted into the transformation law for ϵ_{ijk}. Such tensors are called *pseudo-tensors*.

The *dual vector* of an arbitrary second-order Cartesian tensor T_{ij} is defined by

$$v_i = \epsilon_{ijk} T_{jk} \tag{1.110}$$

which is observed to be the indicial equivalent of $\mathbf{T}_v$, the "vector of the dyadic $\mathbf{T}$", as defined by *(1.15)*.

1.17 MATRICES. MATRIX REPRESENTATION OF CARTESIAN TENSORS

A rectangular array of elements, enclosed by square brackets and subject to certain laws of combination, is called a *matrix*. An $M \times N$ matrix is one having M (horizontal) rows and N (vertical) columns of elements. In the symbol A_{ij}, used to represent the typical element of a matrix, the first subscript denotes the row, the second subscript the column occupied by the element. The matrix itself is designated by enclosing the typical element symbol in square brackets, or alternatively, by the *kernel* letter of the matrix in *script*. For example, the $M \times N$ matrix $\mathcal{A}$, or $[A_{ij}]$ is the array given by

$$\mathcal{A} = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \dots & \dots & \dots & \dots \\ A_{M1} & A_{M2} & \dots & A_{MN} \end{bmatrix} \tag{1.111}$$

A matrix for which $M = N$, is called a *square matrix*. A $1 \times N$ matrix, written $[a_{1k}]$, is called a *row matrix*. An $M \times 1$ matrix, written $[a_{k1}]$, is called a *column matrix*. A matrix having only zeros as elements is called the *zero matrix*. A square matrix with zeros everywhere except on the main diagonal (from A_{11} to A_{NN}) is called a *diagonal matrix*. If the nonzero elements of a diagonal matrix are all *unity*, the matrix is called the *unit* or *identity matrix*. The $N \times M$ matrix $\mathcal{A}^T$, formed by interchanging rows and columns of the $M \times N$ matrix $\mathcal{A}$, is called the *transpose matrix* of $\mathcal{A}$.

Matrices having the same number of rows and columns may be *added* (or subtracted) *element* by *element*. Multiplication of the matrix $[A_{ij}]$ by a scalar λ results in the matrix $[\lambda A_{ij}]$. The product of two matrices, $\mathcal{A}\mathcal{B}$, is defined only if the matrices are *conformable*, i.e. if the *prefactor* matrix $\mathcal{A}$ has the same number of columns as the *postfactor* matrix $\mathcal{B}$ has rows. The product of an $M \times P$ matrix multiplied into a $P \times N$ matrix is an $M \times N$ matrix. Matrix multiplication is usually denoted by simply setting down the matrix symbols in juxtaposition as in

$$\mathcal{A}\mathcal{B} = \mathcal{C} \quad \text{or} \quad [A_{ij}][B_{jk}] = [C_{ik}] \tag{1.112}$$

Matrix multiplication is not, in general, commutative: $\mathcal{A}\mathcal{B} \neq \mathcal{B}\mathcal{A}$.

A square matrix $\mathcal{A}$ whose determinant $|A_{ij}|$ is zero is called a *singular matrix*. The *cofactor* of the element A_{ij} of the square matrix $\mathcal{A}$, denoted here by A_{ij}^*, is defined by

$$A_{ij}^* = (-1)^{i+j} M_{ij} \tag{1.113}$$

in which M_{ij} is the *minor* of A_{ij}; i.e. the determinant of the square array remaining after the row and column of A_{ij} are deleted. The *adjoint* matrix of $\mathcal{A}$ is obtained by replacing each element by its cofactor and then interchanging rows and columns. If a square matrix $\mathcal{A} = [A_{ij}]$ is non-singular, it possesses a unique *inverse matrix* $\mathcal{A}^{-1}$ which is defined as the adjoint matrix of $\mathcal{A}$ divided by the determinant of $\mathcal{A}$. Thus

$$\mathcal{A}^{-1} = \frac{[A_{ji}^*]}{|\mathcal{A}|} \tag{1.114}$$

From the inverse matrix definition (*1.114*) it may be shown that

$$\mathcal{A}^{-1}\mathcal{A} = \mathcal{A}\mathcal{A}^{-1} = \mathcal{I} \tag{1.115}$$

where $\mathcal{I}$ is the *identity matrix*, having ones on the principal diagonal and zeros elsewhere, and so named because of the property

$$\mathcal{I}\mathcal{A} = \mathcal{A}\mathcal{I} = \mathcal{A} \tag{1.116}$$

It is clear, of course, that $\mathcal{I}$ is the matrix representation of δ_{ij}, the Kronecker delta, and of I, the unit dyadic. Any matrix $\mathcal{A}$ for which the condition $\mathcal{A}^T = \mathcal{A}^{-1}$ is satisfied is called an *orthogonal matrix*. Accordingly, if $\mathcal{A}$ is orthogonal,

$$\mathcal{A}^T\mathcal{A} = \mathcal{A}\mathcal{A}^T = \mathcal{I} \tag{1.117}$$

As suggested by the fact that any dyadic may be expressed in the nonion form (*1.53*), and, equivalently, since the components of a second-order tensor may be displayed in the square array (*1.62*), it proves extremely useful to represent second-order tensors (dyadics) by square, 3×3 matrices. A first-order tensor (vector) may be represented by either a 1×3 row matrix, or by a 3×1 column matrix. Although every Cartesian tensor of order two or less (dyadics, vectors, scalars) may be represented by a matrix, not every matrix represents a tensor.

If both matrices in the product $\mathcal{A}\mathcal{B} = \mathcal{C}$ are 3×3 matrices representing second-order tensors, the multiplication is equivalent to the inner product expressed in indicial notation by

$$A_{ij}B_{jk} = C_{ik} \tag{1.118}$$

where the range is three. Expansion of (*1.118*) duplicates the "row by column" multiplication of matrices wherein the elements of the ith row of the prefactor matrix are multiplied in turn by the elements of the kth column of the postfactor matrix, and these products summed to give the element in the ith row and kth column of the product matrix. Several such products occur repeatedly in continuum mechanics and are recorded here in the various notations for reference and comparison.

(*a*) *Vector dot product*

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = \lambda \qquad [a_{1j}][b_{j1}] = [\lambda]$$

$$a_i b_i = b_i a_i = \lambda \qquad [a_1, a_2, a_3]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [a_1 b_1 + a_2 b_2 + a_3 b_3] \tag{1.119}$$

(*b*) *Vector-dyadic dot product*

$$\mathbf{a}\cdot\mathbf{E} = \mathbf{b} \qquad a\mathcal{E} = \mathcal{B}$$

$$a_iE_{ij} = b_j \qquad [a_{1i}][E_{ij}] = [b_{1j}]$$

$$[a_1, a_2, a_3]\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} = \begin{matrix}[a_1E_{11} + a_2E_{21} + a_3E_{31}, \\ a_1E_{12} + a_2E_{22} + a_3E_{32}, \\ a_1E_{13} + a_2E_{23} + a_3E_{33}]\end{matrix} \tag{1.120}$$

(*c*) *Dyadic-vector dot product*

$$\mathbf{E}\cdot\mathbf{a} = \mathbf{c} \qquad \mathcal{E}a = c$$

$$E_{ij}a_j = c_i \qquad [E_{ij}][a_{j1}] = [c_{i1}]$$

$$\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1E_{11} + a_2E_{12} + a_3E_{13} \\ a_1E_{21} + a_2E_{22} + a_3E_{23} \\ a_1E_{31} + a_2E_{32} + a_3E_{33} \end{bmatrix} \tag{1.121}$$

1.18 SYMMETRY OF DYADICS, MATRICES AND TENSORS

According to (*1.36*) (or (*1.37*)), a dyadic $\mathbf{D}$ is said to be symmetric (anti-symmetric) if it is equal to (the negative of) its conjugate $\mathbf{D}_c$. Similarly the second-order tensor D_{ij} is *symmetric* if

$$D_{ij} = D_{ji} \tag{1.122}$$

and is *anti-symmetric*, or *skew-symmetric*, if

$$D_{ij} = -D_{ji} \tag{1.123}$$

Therefore the decomposition of D_{ij} analogous to (*1.38*) is

$$D_{ij} = \tfrac{1}{2}(D_{ij} + D_{ji}) + \tfrac{1}{2}(D_{ij} - D_{ji}) \tag{1.124}$$

or, in an equivalent abbreviated form often employed,

$$D_{ij} = D_{(ij)} + D_{[ij]} \tag{1.125}$$

where parentheses around the indices denote the symmetric part of D_{ij}, and square brackets on the indices denote the anti-symmetric part.

Since the interchange of indices of a second-order tensor is equivalent to the interchange of rows and columns in its matrix representation, a square matrix $\mathcal{A}$ is symmetric if it is equal to its transpose $\mathcal{A}^T$. Consequently a symmetric 3×3 matrix has only six independent components as illustrated by

$$\mathcal{A} = \mathcal{A}^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \tag{1.126}$$

An anti-symmetric matrix is one that equals the *negative* of its transpose. Consequently a 3×3 anti-symmetric matrix $\mathcal{B}$ has zeros on the main diagonal, and therefore only three independent components as illustrated by

$$\mathcal{B} = -\mathcal{B}^T = \begin{bmatrix} 0 & B_{12} & B_{13} \\ -B_{12} & 0 & B_{23} \\ -B_{13} & -B_{23} & 0 \end{bmatrix} \tag{1.127}$$

Symmetry properties may be extended to tensors of higher order than two. In general, an arbitrary tensor is said to be symmetric with respect to a pair of indices if the value of the typical component is unchanged by interchanging these two indices. A tensor is anti-symmetric in a pair of indices if an interchange of these indices leads to a change of sign without a change of absolute value in the component. Examples of symmetry properties in higher-order tensors are

(a) $R_{ijkm} = R_{ikjm}$ (symmetric in k and j)

(b) $\epsilon_{ijk} = -\epsilon_{kji}$ (anti-symmetric in k and i)

(c) $G_{ijkm} = G_{jimk}$ (symmetric in i and j; k and m)

(d) $\beta_{ijk} = \beta_{ikj} = \beta_{kji} = \beta_{jik}$ (symmetric in all indices)

1.19 PRINCIPAL VALUES AND PRINCIPAL DIRECTIONS OF SYMMETRIC SECOND-ORDER TENSORS

In the following analysis, only symmetric tensors with real components are considered. This simplifies the mathematics somewhat, and since the important tensors of continuum mechanics are usually symmetric there is little sacrifice in this restriction.

For every symmetric tensor T_{ij}, defined at some point in space, there is associated with each direction (specified by the unit normal n_i) at that point, a vector given by the inner product

$$v_i = T_{ij}n_j \tag{1.128}$$

Here T_{ij} may be envisioned as a linear vector operator which produces the vector v_i conjugate to the direction n_i. If the direction is one for which v_i is parallel to n_i, the inner product may be expressed as a scalar multiple of n_i. For this case,

$$T_{ij}n_j = \lambda n_i \tag{1.129}$$

and the direction n_i is called a *principal direction*, or *principal axis* of T_{ij}. With the help of the identity $n_i = \delta_{ij}n_j$, (*1.129*) can be put in the form

$$(T_{ij} - \lambda\delta_{ij})n_j = 0 \tag{1.130}$$

which represents a system of three equations for the four unknowns, n_i and λ, associated with each principal direction. In expanded form, the system to be solved is

$$\begin{aligned}(T_{11} - \lambda)n_1 + T_{12}n_2 + T_{13}n_3 &= 0\\ T_{21}n_1 + (T_{22} - \lambda)n_2 + T_{23}n_3 &= 0\\ T_{31}n_1 + T_{32}n_2 + (T_{33} - \lambda)n_3 &= 0\end{aligned} \tag{1.131}$$

Note first that for every λ, the trivial solution $n_i = 0$ satisfies the equations. The purpose here, however, is to obtain non-trivial solutions. Also, from the homogeneity of the system (*1.131*) it follows that no loss of generality is incurred by restricting attention to solutions for which $n_in_i = 1$, and this condition is imposed from now on.

For (*1.130*) or, equivalently, (*1.131*) to have a non-trivial solution, the determinant of coefficients must be zero, that is,

$$|T_{ij} - \lambda\delta_{ij}| = 0 \tag{1.132}$$

Expansion of this determinant leads to a cubic polynomial in λ, namely,

$$\lambda^3 - \mathrm{I}_T\lambda^2 + \mathrm{II}_T\lambda - \mathrm{III}_T = 0 \tag{1.133}$$

which is known as the *characteristic equation* of T_{ij}, and for which the scalar coefficients,

$$\mathrm{I}_T = T_{ii} = \operatorname{tr} T_{ij} \text{ (trace of } T_{ij}) \tag{1.134}$$

$$\mathrm{II}_T = \tfrac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ij}) \tag{1.135}$$

$$\mathrm{III}_T = |T_{ij}| = \det T_{ij} \tag{1.136}$$

are called the first, second and third *invariants*, respectively, of T_{ij}. The three roots of the cubic (*1.133*), labeled $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}$, are called the *principal values* of T_{ij}. For a symmetric tensor with real components, the principal values are real; and if these values are distinct, the three principal directions are mutually orthogonal. When referred to principal axes, both the tensor array and its matrix appear in diagonal form. Thus

$$\mathbf{T} = \begin{pmatrix} \lambda_{(1)} & 0 & 0 \\ 0 & \lambda_{(2)} & 0 \\ 0 & 0 & \lambda_{(3)} \end{pmatrix} \quad \text{or} \quad \mathcal{T} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 \\ 0 & \lambda_{(2)} & 0 \\ 0 & 0 & \lambda_{(3)} \end{bmatrix} \tag{1.137}$$

If $\lambda_{(1)} = \lambda_{(2)}$, the tensor has a diagonal form which is independent of the choice of $\lambda_{(1)}$ and $\lambda_{(2)}$ axes, once the principal axis associated with $\lambda_{(3)}$ has been established. If all principal values are equal, any direction is a principal direction. If the principal values are ordered, it is customary to write them as $\lambda_{(I)}, \lambda_{(II)}, \lambda_{(III)}$ and to display the ordering as in $\lambda_{(I)} > \lambda_{(II)} > \lambda_{(III)}$.

For principal axes labeled $Ox_1^*x_2^*x_3^*$, the transformation from $Ox_1x_2x_3$ axes is given by the elements of the table

	x_1	x_2	x_3
x_1^*	$a_{11} = n_1^{(1)}$	$a_{12} = n_2^{(1)}$	$a_{13} = n_3^{(1)}$
x_2^*	$a_{21} = n_1^{(2)}$	$a_{22} = n_2^{(2)}$	$a_{23} = n_3^{(2)}$
x_3^*	$a_{31} = n_1^{(3)}$	$a_{32} = n_2^{(3)}$	$a_{33} = n_3^{(3)}$

in which $n_i^{(j)}$ are the direction cosines of the jth principal direction.

1.20 POWERS OF SECOND-ORDER TENSORS. HAMILTON-CAYLEY EQUATION

By direct matrix multiplication, the square of the tensor T_{ij} is given as the inner product $T_{ik}T_{kj}$; the cube as $T_{ik}T_{km}T_{mj}$; etc. Therefore with T_{ij} written in the diagonal form (*1.137*), the nth power of the tensor is given by

$$(\mathbf{T})^n = \begin{pmatrix} \lambda_{(1)}^n & 0 & 0 \\ 0 & \lambda_{(2)}^n & 0 \\ 0 & 0 & \lambda_{(3)}^n \end{pmatrix} \quad \text{or} \quad \mathcal{T}^n = \begin{bmatrix} \lambda_{(1)}^n & 0 & 0 \\ 0 & \lambda_{(2)}^n & 0 \\ 0 & 0 & \lambda_{(3)}^n \end{bmatrix} \tag{1.138}$$

A comparison of (*1.138*) and (*1.137*) indicates that T_{ij} and all its integer powers have the same principal axes.

Since each of the principal values satisfies (*1.133*), and because of the diagonal matrix form of $\mathcal{T}^n$ given by (*1.138*), the tensor itself will satisfy (*1.133*). Thus

$$\mathcal{T}^3 - \mathrm{I}_T\mathcal{T}^2 + \mathrm{II}_T\mathcal{T} - \mathrm{III}_T\mathcal{I} = 0 \tag{1.139}$$

in which $\mathcal{I}$ is the identity matrix. This equation is called the *Hamilton-Cayley equation.* Matrix multiplication of each term in (*1.139*) by $\mathcal{T}$ produces the equation,

$$\mathcal{T}^4 = \mathrm{I}_T\mathcal{T}^3 - \mathrm{II}_T\mathcal{T}^2 + \mathrm{III}_T\mathcal{T} \tag{1.140}$$

Combining (*1.140*) and (*1.139*) by direct substitution,

$$\mathcal{T}^4 = (\mathrm{I}_T^2 - \mathrm{II}_T)\mathcal{T}^2 + (\mathrm{III}_T - \mathrm{I}_T\mathrm{II}_T)\mathcal{T} + \mathrm{I}_T\mathrm{III}_T\mathcal{I} \qquad (1.141)$$

Continuation of this procedure yields the positive powers of $\mathcal{T}$ as linear combinations of $\mathcal{T}^2, \mathcal{T}$ and $\mathcal{I}$.

1.21 TENSOR FIELDS. DERIVATIVES OF TENSORS

A *tensor field* assigns a tensor $\mathbf{T}(\mathbf{x}, t)$ to every pair $(\mathbf{x}, t)$ where the position vector $\mathbf{x}$ varies over a particular region of space and t varies over a particular interval of time. The tensor field is said to be continuous (or differentiable) if the components of $\mathbf{T}(\mathbf{x}, t)$ are continuous (or differentiable) functions of $\mathbf{x}$ and t. If the components are functions of $\mathbf{x}$ only, the tensor field is said to be *steady*.

With respect to a rectangular Cartesian coordinate system, for which the position vector of an arbitrary point is

$$\mathbf{x} = x_i\hat{\mathbf{e}}_i \qquad (1.142)$$

tensor fields of various orders are represented in indicial and symbolic notation as follows,

(*a*) scalar field: $\phi = \phi(x_i, t)$ or $\phi = \phi(\mathbf{x}, t)$ (*1.143*)

(*b*) vector field: $v_i = v_i(\mathbf{x}, t)$ or $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ (*1.144*)

(*c*) second-order tensor field:

$$T_{ij} = T_{ij}(\mathbf{x}, t) \quad \text{or} \quad \mathbf{T} = \mathbf{T}(\mathbf{x}, t) \qquad (1.145)$$

Coordinate differentiation of tensor components with respect to x_i is expressed by the differential operator $\partial/\partial x_i$, or briefly in indicial form by ∂_i, indicating an operator of tensor rank one. In symbolic notation, the corresponding symbol is the well-known differential vector operator ∇, pronounced *del* and written explicitly

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} = \hat{\mathbf{e}}_i\partial_i \qquad (1.146)$$

Frequently, partial differentiation with respect to the variable x_i is represented by the *comma-subscript convention* as illustrated by the following examples.

(*a*) $\dfrac{\partial \phi}{\partial x_i} = \phi_{,i}$ (*d*) $\dfrac{\partial^2 v_i}{\partial x_j\,\partial x_k} = v_{i,jk}$

(*b*) $\dfrac{\partial v_i}{\partial x_i} = v_{i,i}$ (*e*) $\dfrac{\partial T_{ij}}{\partial x_k} = T_{ij,k}$

(*c*) $\dfrac{\partial v_i}{\partial x_j} = v_{i,j}$ (*f*) $\dfrac{\partial^2 T_{ij}}{\partial x_k\,\partial x_m} = T_{ij,km}$

From these examples it is seen that the operator ∂_i produces a tensor of order one higher if i remains a free index ((*a*) and (*c*) above), and a tensor of order one lower if i becomes a dummy index ((*b*) above) in the derivative.

Several important differential operators appear often in continuum mechanics and are given here for reference.

$$\operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x_i}\hat{\mathbf{e}}_i \quad \text{or} \quad \partial_i\phi = \phi_{,i} \qquad (1.147)$$

$$\operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} \quad \text{or} \quad \partial_i v_i = v_{i,i} \qquad (1.148)$$

$$\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} \quad \text{or} \quad \epsilon_{ijk}\partial_j v_k = \epsilon_{ijk} v_{k,j} \qquad (1.149)$$

$$\nabla^2\phi = \nabla\cdot\nabla\phi \quad \text{or} \quad \partial_{ii}\phi = \phi_{,ii} \qquad (1.150)$$

1.22 LINE INTEGRALS. STOKES' THEOREM

In a given region of space the vector function of position, $\mathbf{F} = \mathbf{F}(\mathbf{x})$, is defined at every point of the piecewise smooth curve C shown in Fig. 1-10. If the *differential tangent vector* to the curve at the arbitrary point P is $d\mathbf{x}$, the integral

$$\int_C \mathbf{F}\cdot d\mathbf{x} \quad \equiv \quad \int_{\mathbf{x}_A}^{\mathbf{x}_B} \mathbf{F}\cdot d\mathbf{x} \tag{1.151}$$

taken along the curve from A to B is known as the *line integral* of F along C. In the indicial notation, (*1.151*) becomes

$$\int_C F_i\,dx_i \quad \equiv \quad \int_{(x_i)_A}^{(x_i)_B} F_i\,dx_i \tag{1.152}$$

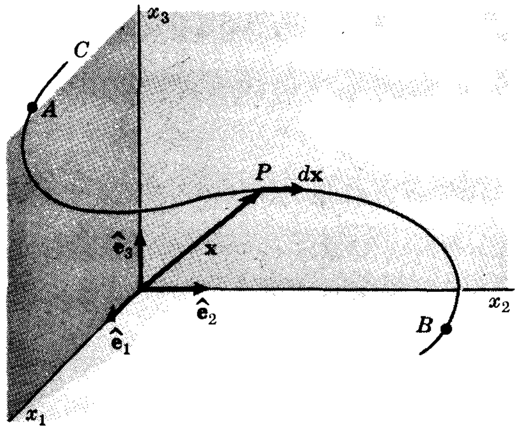


Fig. 1-10

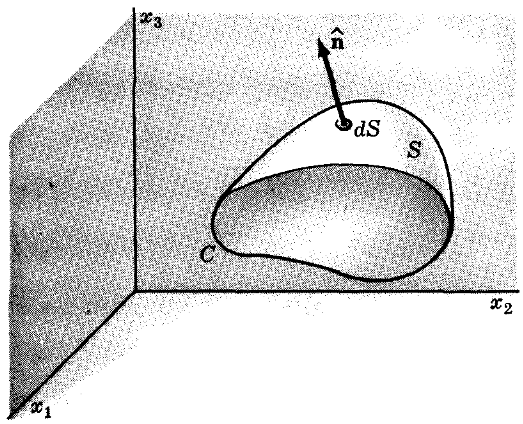


Fig. 1-11

Stokes' theorem says that the line integral of $\mathbf{F}$ taken around a closed reducible curve C, as pictured in Fig. 1-11, may be expressed in terms of an integral over any two-sided surface S which has C as its boundary. Explicitly,

$$\oint_C \mathbf{F}\cdot d\mathbf{x} \quad = \quad \int_S \hat{\mathbf{n}}\cdot(\nabla\times\mathbf{F})\,dS \tag{1.153}$$

in which $\hat{\mathbf{n}}$ is the unit normal on the positive side of S, and dS is the differential element of surface as shown by the figure. In the indicial notation, (*1.153*) is written

$$\oint_C F_i\,dx_i \quad = \quad \int_S n_i\epsilon_{ijk}F_{k,j}\,dS \tag{1.154}$$

1.23 THE DIVERGENCE THEOREM OF GAUSS

The *divergence theorem of Gauss* relates a volume integral to a surface integral. In its traditional form the theorem says that for the vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$,

$$\int_V \operatorname{div}\mathbf{v}\,dV \quad = \quad \int_S \hat{\mathbf{n}}\cdot\mathbf{v}\,dS \tag{1.155}$$

where $\hat{\mathbf{n}}$ is the outward unit normal to the bounding surface S, of the volume V in which the vector field is defined. In the indicial notation, (*1.155*) is written

$$\int_V v_{i,i}\,dV \quad = \quad \int_S v_i n_i\,dS \tag{1.156}$$

The divergence theorem of Gauss as expressed by (*1.156*) may be generalized to incorporate a tensor field of any order. Thus for the arbitrary tensor field $T_{ijk\ldots}$ the theorem is written

$$\int_V T_{ijk\ldots,p}\,dV \quad = \quad \int_S T_{ijk\ldots}n_p\,dS \tag{1.157}$$

Solved Problems

ALGEBRA OF VECTORS AND DYADICS (Sec. 1.1-1.8)

1.1. Determine in rectangular Cartesian form the unit vector which is (a) parallel to the vector $\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, (b) along the line joining points $P(1,0,3)$ and $Q(0,2,1)$.

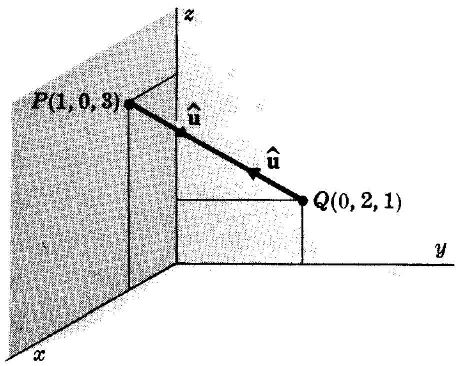


Fig. 1-12

(a) $|\mathbf{v}| = v = \sqrt{(2)^2 + (3)^2 + (-6)^2} = 7$

$\hat{\mathbf{v}} = \mathbf{v}/v = (2/7)\hat{\mathbf{i}} + (3/7)\hat{\mathbf{j}} - (6/7)\hat{\mathbf{k}}$

(b) The vector extending from P to Q is

$$\mathbf{u} = (0-1)\hat{\mathbf{i}} + (2-0)\hat{\mathbf{j}} + (1-3)\hat{\mathbf{k}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$u = \sqrt{(-1)^2 + (2)^2 + (-2)^2} = 3$$

Thus $\hat{\mathbf{u}} = -(1/3)\hat{\mathbf{i}} + (2/3)\hat{\mathbf{j}} - (2/3)\hat{\mathbf{k}}$ directed from P to Q

or $\hat{\mathbf{u}} = (1/3)\hat{\mathbf{i}} - (2/3)\hat{\mathbf{j}} + (2/3)\hat{\mathbf{k}}$ directed from Q to P

1.2. Prove that the vector $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ is normal to the plane whose equation is $ax + by + cz = \lambda$.

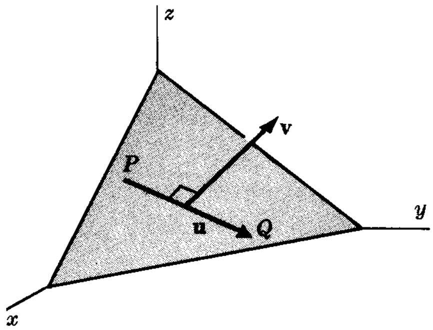


Fig. 1-13

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points in the plane. Then $ax_1 + by_1 + cz_1 = \lambda$ and $ax_2 + by_2 + cz_2 = \lambda$ and the vector joining these points is $\mathbf{u} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}$. The projection of $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\frac{\mathbf{u}\cdot\mathbf{v}}{u} = \frac{1}{u}[(x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}]\cdot[a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}]$$
$$= \frac{1}{u}(ax_2 + by_2 + cz_2 - ax_1 - by_1 - cz_1) = \frac{\lambda - \lambda}{u} = 0$$

Since $\mathbf{u}$ is any vector in the plane, $\mathbf{v}$ is $\perp$ to the plane.

1.3. If $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is the vector extending from the origin to the arbitrary point $P(x,y,z)$ and $\mathbf{d} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ is a constant vector, show that $(\mathbf{r} - \mathbf{d})\cdot\mathbf{r} = 0$ is the vector equation of a sphere.

Expanding the indicated dot product,

$$(\mathbf{r} - \mathbf{d})\cdot\mathbf{r} = [(x-a)\hat{\mathbf{i}} + (y-b)\hat{\mathbf{j}} + (z-c)\hat{\mathbf{k}}]\cdot[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}]$$
$$= x^2 + y^2 + z^2 - ax - by - cz = 0$$

Adding $d^2/4 = (a^2 + b^2 + c^2)/4$ to each side of this equation gives the desired equation

$$(x - a/2)^2 + (y - b/2)^2 + (z - c/2)^2 = (d/2)^2$$

which is the equation of the sphere centered at $\mathbf{d}/2$ with radius $d/2$.

1.4. Prove that $[\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}]\mathbf{r} = (\mathbf{a}\cdot\mathbf{r})\mathbf{b}\times\mathbf{c} + (\mathbf{b}\cdot\mathbf{r})\mathbf{c}\times\mathbf{a} + (\mathbf{c}\cdot\mathbf{r})\mathbf{a}\times\mathbf{b}$.

Consider the product $\mathbf{a}\times[(\mathbf{b}\times\mathbf{c})\times\mathbf{r}]$. By direct expansion of the cross product in brackets,

$$\mathbf{a}\times[(\mathbf{b}\times\mathbf{c})\times\mathbf{r}] = \mathbf{a}\times[(\mathbf{b}\cdot\mathbf{r})\mathbf{c} - (\mathbf{c}\cdot\mathbf{r})\mathbf{b}] = -(\mathbf{b}\cdot\mathbf{r})\mathbf{c}\times\mathbf{a} - (\mathbf{c}\cdot\mathbf{r})\mathbf{a}\times\mathbf{b}$$

Also, setting $\mathbf{b}\times\mathbf{c} = \mathbf{v}$,

$$\mathbf{a}\times[(\mathbf{b}\times\mathbf{c})\times\mathbf{r}] = \mathbf{a}\times(\mathbf{v}\times\mathbf{r}) = (\mathbf{a}\cdot\mathbf{r})\mathbf{b}\times\mathbf{c} - (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})\mathbf{r}$$

Thus $-(\mathbf{b}\cdot\mathbf{r})\mathbf{c}\times\mathbf{a} - (\mathbf{c}\cdot\mathbf{r})\mathbf{a}\times\mathbf{b} = (\mathbf{a}\cdot\mathbf{r})\mathbf{b}\times\mathbf{c} - (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})\mathbf{r}$ and so

$$(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})\mathbf{r} = (\mathbf{a}\cdot\mathbf{r})\mathbf{b}\times\mathbf{c} + (\mathbf{b}\cdot\mathbf{r})\mathbf{c}\times\mathbf{a} + (\mathbf{c}\cdot\mathbf{r})\mathbf{a}\times\mathbf{b}$$

This identity is useful in specifying the displacement of a rigid body in terms of three arbitrary points of the body.

1.5. Show that if the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are linearly dependent, $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = 0$. Check the linear dependence or independence of the basis

$$\mathbf{u} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$
$$\mathbf{v} = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$$
$$\mathbf{w} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent if there exist constants λ, μ and ν, not all zero, such that $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = 0$. The component scalar equations of this vector equation are

$$\lambda a_x + \mu b_x + \nu c_x = 0$$
$$\lambda a_y + \mu b_y + \nu c_y = 0$$
$$\lambda a_z + \mu b_z + \nu c_z = 0$$

This set has a nonzero solution for λ, μ and ν provided the determinant of coefficients vanishes,

$$\begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix} = 0$$

which is equivalent to $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = 0$. For the proposed basis $\mathbf{u}, \mathbf{v}, \mathbf{w}$,

$$\begin{vmatrix} 3 & 1 & -2 \\ 4 & -1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

Hence the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent, and indeed $\mathbf{v} = \mathbf{u} + \mathbf{w}$.

1.6. Show that any dyadic of N terms may be reduced to a dyadic of three terms in a form having the base vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ as (*a*) antecedents, (*b*) consequents.

Let $\mathbf{D} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_N\mathbf{b}_N = \mathbf{a}_i\mathbf{b}_i$ $(i = 1, 2, \ldots, N)$.

(*a*) In terms of base vectors, $\mathbf{a}_i = a_{1i}\hat{\mathbf{e}}_1 + a_{2i}\hat{\mathbf{e}}_2 + a_{3i}\hat{\mathbf{e}}_3 = a_{ji}\hat{\mathbf{e}}_j$ and so $\mathbf{D} = a_{ji}\hat{\mathbf{e}}_j\mathbf{b}_i = \hat{\mathbf{e}}_j(a_{ji}\mathbf{b}_i) = \hat{\mathbf{e}}_j\mathbf{c}_j$ with $j = 1, 2, 3$.

(*b*) Likewise setting $\mathbf{b}_i = b_{ji}\hat{\mathbf{e}}_j$ it follows that $\mathbf{D} = \mathbf{a}_i b_{ji}\hat{\mathbf{e}}_j = (b_{ji}\mathbf{a}_i)\hat{\mathbf{e}}_j = \mathbf{g}_j\hat{\mathbf{e}}_j$ where $j = 1, 2, 3$.

1.7. For the arbitrary dyadic $\mathbf{D}$ and vector $\mathbf{v}$, show that $\mathbf{D}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{D}_c$.

Let $\mathbf{D} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_N\mathbf{b}_N$. Then

$$\mathbf{D}\cdot\mathbf{v} = \mathbf{a}_1(\mathbf{b}_1\cdot\mathbf{v}) + \mathbf{a}_2(\mathbf{b}_2\cdot\mathbf{v}) + \cdots + \mathbf{a}_N(\mathbf{b}_N\cdot\mathbf{v})$$
$$= (\mathbf{v}\cdot\mathbf{b}_1)\mathbf{a}_1 + (\mathbf{v}\cdot\mathbf{b}_2)\mathbf{a}_2 + \cdots + (\mathbf{v}\cdot\mathbf{b}_N)\mathbf{a}_N = \mathbf{v}\cdot\mathbf{D}_c$$

1.8. Prove that $(\mathbf{D}_c\cdot\mathbf{D})_c = \mathbf{D}_c\cdot\mathbf{D}$.

From *(1.71)*, $\mathbf{D} = D_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$ and $\mathbf{D}_c = D_{ji}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$. Therefore

$$\mathbf{D}_c\cdot\mathbf{D} = D_{ji}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\cdot D_{pq}\hat{\mathbf{e}}_p\hat{\mathbf{e}}_q = D_{ji}D_{pq}(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_p)\hat{\mathbf{e}}_i\hat{\mathbf{e}}_q$$

and $$(\mathbf{D}_c\cdot\mathbf{D})_c = D_{ji}D_{pq}(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_p)\hat{\mathbf{e}}_q\hat{\mathbf{e}}_i = D_{pq}\hat{\mathbf{e}}_q(\hat{\mathbf{e}}_p\cdot\hat{\mathbf{e}}_j)\hat{\mathbf{e}}_i D_{ji} = D_{pq}\hat{\mathbf{e}}_q\hat{\mathbf{e}}_p\cdot D_{ji}\hat{\mathbf{e}}_j\hat{\mathbf{e}}_i = \mathbf{D}_c\cdot\mathbf{D}$$

1.9. Show that $(\mathbf{D}\times\mathbf{v})_c = -\mathbf{v}\times\mathbf{D}_c$.

$$\mathbf{D}\times\mathbf{v} = \mathbf{a}_1(\mathbf{b}_1\times\mathbf{v}) + \mathbf{a}_2(\mathbf{b}_2\times\mathbf{v}) + \cdots + \mathbf{a}_N(\mathbf{b}_N\times\mathbf{v})$$

$$(\mathbf{D}\times\mathbf{v})_c = (\mathbf{b}_1\times\mathbf{v})\mathbf{a}_1 + (\mathbf{b}_2\times\mathbf{v})\mathbf{a}_2 + \cdots + (\mathbf{b}_N\times\mathbf{v})\mathbf{a}_N$$
$$= -(\mathbf{v}\times\mathbf{b}_1)\mathbf{a}_1 - (\mathbf{v}\times\mathbf{b}_2)\mathbf{a}_2 - \cdots - (\mathbf{v}\times\mathbf{b}_N)\mathbf{a}_N = -\mathbf{v}\times\mathbf{D}_c$$

1.10. If $\mathbf{D} = a\hat{\mathbf{i}}\hat{\mathbf{i}} + b\hat{\mathbf{j}}\hat{\mathbf{j}} + c\hat{\mathbf{k}}\hat{\mathbf{k}}$ and $\mathbf{r}$ is the position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, show that $\mathbf{r}\cdot\mathbf{D}\cdot\mathbf{r} = 1$ represents the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

$$\mathbf{r}\cdot\mathbf{D}\cdot\mathbf{r} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(a\hat{\mathbf{i}}\hat{\mathbf{i}} + b\hat{\mathbf{j}}\hat{\mathbf{j}} + c\hat{\mathbf{k}}\hat{\mathbf{k}})\cdot(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$
$$= (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot(ax\hat{\mathbf{i}} + by\hat{\mathbf{j}} + cz\hat{\mathbf{k}}) = ax^2 + by^2 + cz^2 = 1$$

1.11. For the dyadics $\mathbf{D} = 3\hat{\mathbf{i}}\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\hat{\mathbf{j}} - \hat{\mathbf{j}}\hat{\mathbf{k}} + 5\hat{\mathbf{k}}\hat{\mathbf{k}}$ and $\mathbf{F} = 4\hat{\mathbf{i}}\hat{\mathbf{k}} + 6\hat{\mathbf{j}}\hat{\mathbf{j}} - 3\hat{\mathbf{k}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}}$, compute and compare the double dot products $\mathbf{D}:\mathbf{F}$ and $\mathbf{D}\cdot\cdot\mathbf{F}$.

From the definition $\mathbf{ab}:\mathbf{cd} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})$ it is seen that $\mathbf{D}:\mathbf{F} = 12 + 5 = 17$. Also, from $\mathbf{ab}\cdot\cdot\,\mathbf{cd} = (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$ it follows that $\mathbf{D}\cdot\cdot\mathbf{F} = 12 + 3 + 5 = 20$.

1.12. Determine the dyadics $\mathbf{G} = \mathbf{D}\cdot\mathbf{F}$ and $\mathbf{H} = \mathbf{F}\cdot\mathbf{D}$ if $\mathbf{D}$ and $\mathbf{F}$ are the dyadics given in Problem 1.11.

From the definition $\mathbf{ab}\cdot\mathbf{cd} = (\mathbf{b}\cdot\mathbf{c})\mathbf{ad}$,

$$\mathbf{G} = (3\hat{\mathbf{i}}\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\hat{\mathbf{j}} - \hat{\mathbf{j}}\hat{\mathbf{k}} + 5\hat{\mathbf{k}}\hat{\mathbf{k}})\cdot(4\hat{\mathbf{i}}\hat{\mathbf{k}} + 6\hat{\mathbf{j}}\hat{\mathbf{j}} - 3\hat{\mathbf{k}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}})$$
$$= 12\hat{\mathbf{i}}\hat{\mathbf{k}} + 12\hat{\mathbf{j}}\hat{\mathbf{j}} + 3\hat{\mathbf{j}}\hat{\mathbf{j}} - \hat{\mathbf{j}}\hat{\mathbf{k}} - 15\hat{\mathbf{k}}\hat{\mathbf{j}} + 5\hat{\mathbf{k}}\hat{\mathbf{k}}$$

Similarly,
$$\mathbf{H} = (4\hat{\mathbf{i}}\hat{\mathbf{k}} + 6\hat{\mathbf{j}}\hat{\mathbf{j}} - 3\hat{\mathbf{k}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}})\cdot(3\hat{\mathbf{i}}\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\hat{\mathbf{j}} - \hat{\mathbf{j}}\hat{\mathbf{k}} + 5\hat{\mathbf{k}}\hat{\mathbf{k}})$$
$$= 20\hat{\mathbf{i}}\hat{\mathbf{k}} + 12\hat{\mathbf{j}}\hat{\mathbf{j}} - 6\hat{\mathbf{j}}\hat{\mathbf{k}} - 6\hat{\mathbf{k}}\hat{\mathbf{j}} + 8\hat{\mathbf{k}}\hat{\mathbf{k}}$$

1.13. Show directly from the nonion form of the dyadic $\mathbf{D}$ that $\mathbf{D} = (\mathbf{D}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{D}\cdot\hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{D}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}$ and also $\hat{\mathbf{i}}\cdot\mathbf{D}\cdot\hat{\mathbf{i}} = D_{xx}$, $\hat{\mathbf{i}}\cdot\mathbf{D}\cdot\hat{\mathbf{j}} = D_{xy}$, etc.

Writing $\mathbf{D}$ in nonion form and regrouping terms,

$$\mathbf{D} = (D_{xx}\hat{\mathbf{i}} + D_{yx}\hat{\mathbf{j}} + D_{zx}\hat{\mathbf{k}})\hat{\mathbf{i}} + (D_{xy}\hat{\mathbf{i}} + D_{yy}\hat{\mathbf{j}} + D_{zy}\hat{\mathbf{k}})\hat{\mathbf{j}} + (D_{xz}\hat{\mathbf{i}} + D_{yz}\hat{\mathbf{j}} + D_{zz}\hat{\mathbf{k}})\hat{\mathbf{k}}$$
$$= \mathbf{d}_1\hat{\mathbf{i}} + \mathbf{d}_2\hat{\mathbf{j}} + \mathbf{d}_3\hat{\mathbf{k}} = (\mathbf{D}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{D}\cdot\hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{D}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}$$

Also now
$$\hat{\mathbf{i}}\cdot\mathbf{d}_1 = \hat{\mathbf{i}}\cdot(\mathbf{D}\cdot\hat{\mathbf{i}}) = \hat{\mathbf{i}}\cdot(D_{xx}\hat{\mathbf{i}} + D_{yx}\hat{\mathbf{j}} + D_{zx}\hat{\mathbf{k}}) = D_{xx}$$
$$\hat{\mathbf{j}}\cdot\mathbf{d}_1 = \hat{\mathbf{j}}\cdot\mathbf{D}\cdot\hat{\mathbf{i}} = D_{yx}, \qquad \hat{\mathbf{j}}\cdot\mathbf{d}_2 = \hat{\mathbf{j}}\cdot\mathbf{D}\cdot\hat{\mathbf{j}} = D_{yy}, \text{ etc.}$$

1.14. For an antisymmetric dyadic $\mathbf{A}$ and the arbitrary vector $\mathbf{b}$, show that $2\mathbf{b}\cdot\mathbf{A} = \mathbf{A}_v\times\mathbf{b}$.

From Problem 1.6(*a*), $\mathbf{A} = \hat{\mathbf{e}}_1\mathbf{c}_1 + \hat{\mathbf{e}}_2\mathbf{c}_2 + \hat{\mathbf{e}}_3\mathbf{c}_3$; and because it is antisymmetric, $2\mathbf{A} = (\mathbf{A} - \mathbf{A}_c)$ or

$$2\mathbf{A} = (\hat{\mathbf{e}}_1\mathbf{c}_1 + \hat{\mathbf{e}}_2\mathbf{c}_2 + \hat{\mathbf{e}}_3\mathbf{c}_3 - \mathbf{c}_1\hat{\mathbf{e}}_1 - \mathbf{c}_2\hat{\mathbf{e}}_2 - \mathbf{c}_3\hat{\mathbf{e}}_3)$$
$$= (\hat{\mathbf{e}}_1\mathbf{c}_1 - \mathbf{c}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\mathbf{c}_2 - \mathbf{c}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\mathbf{c}_3 - \mathbf{c}_3\hat{\mathbf{e}}_3)$$

and so
$$2\mathbf{b}\cdot\mathbf{A} = [(\mathbf{b}\cdot\hat{\mathbf{e}}_1)\mathbf{c}_1 - (\mathbf{b}\cdot\mathbf{c}_1)\hat{\mathbf{e}}_1] + [(\mathbf{b}\cdot\hat{\mathbf{e}}_2)\mathbf{c}_2 - (\mathbf{b}\cdot\mathbf{c}_2)\hat{\mathbf{e}}_2] + [(\mathbf{b}\cdot\hat{\mathbf{e}}_3)\mathbf{c}_3 - (\mathbf{b}\cdot\mathbf{c}_3)\hat{\mathbf{e}}_3]$$
$$= [(\hat{\mathbf{e}}_1\times\mathbf{c}_1)\times\mathbf{b} + (\hat{\mathbf{e}}_2\times\mathbf{c}_2)\times\mathbf{b} + (\hat{\mathbf{e}}_3\times\mathbf{c}_3)\times\mathbf{b}] = (\mathbf{A}_v\times\mathbf{b})$$

1.15. If $\mathbf{D} = 6\hat{\mathbf{i}}\hat{\mathbf{i}} + 3\hat{\mathbf{i}}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\hat{\mathbf{k}}$ and $\mathbf{u} = 2\hat{\mathbf{i}} + \hat{\mathbf{k}}$, $\mathbf{v} = 5\hat{\mathbf{j}}$, show by direct calculation that $\mathbf{D}\cdot(\mathbf{u}\times\mathbf{v}) = (\mathbf{D}\times\mathbf{u})\cdot\mathbf{v}$.

Since $\mathbf{u}\times\mathbf{v} = (2\hat{\mathbf{i}} + \hat{\mathbf{k}})\times 5\hat{\mathbf{j}} = 10\hat{\mathbf{k}} - 5\hat{\mathbf{i}}$,

$$\mathbf{D}\cdot(\mathbf{u}\times\mathbf{v}) = (6\hat{\mathbf{i}}\hat{\mathbf{i}} + 3\hat{\mathbf{i}}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\hat{\mathbf{k}})\cdot(-5\hat{\mathbf{i}} + 10\hat{\mathbf{k}}) = -30\hat{\mathbf{i}} + 40\hat{\mathbf{k}}$$

Next,
$$\mathbf{D}\times\mathbf{u} = (6\hat{\mathbf{i}}\hat{\mathbf{i}} + 3\hat{\mathbf{i}}\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\hat{\mathbf{k}})\times(2\hat{\mathbf{i}} + \hat{\mathbf{k}}) = -6\hat{\mathbf{i}}\hat{\mathbf{k}} + 8\hat{\mathbf{k}}\hat{\mathbf{j}} - 6\hat{\mathbf{i}}\hat{\mathbf{j}} + 3\hat{\mathbf{i}}\hat{\mathbf{i}}$$

and
$$(\mathbf{D}\times\mathbf{u})\cdot\mathbf{v} = (3\hat{\mathbf{i}}\hat{\mathbf{i}} - 6\hat{\mathbf{i}}\hat{\mathbf{j}} - 6\hat{\mathbf{i}}\hat{\mathbf{k}} + 8\hat{\mathbf{k}}\hat{\mathbf{j}})\cdot 5\hat{\mathbf{j}} = -30\hat{\mathbf{i}} + 40\hat{\mathbf{k}}$$

1.16. Considering the dyadic

$$\mathbf{D} = 3\hat{\mathbf{i}}\hat{\mathbf{i}} - 4\hat{\mathbf{i}}\hat{\mathbf{j}} + 2\hat{\mathbf{j}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}}$$

as a linear vector operator, determine the vector $\mathbf{r}'$ produced when $\mathbf{D}$ operates on $\mathbf{r} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$.

$$\begin{aligned} \mathbf{r}' &= \mathbf{D}\cdot\mathbf{r} \\ &= 12\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 8\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}} \\ &= 4\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 5\hat{\mathbf{k}} \end{aligned}$$

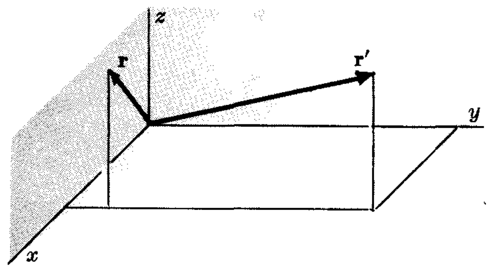


Fig. 1-14

1.17. Determine the dyadic $\mathbf{D}$ which serves as a linear vector operator for the vector function $\mathbf{a} = \mathbf{f}(\mathbf{b}) = \mathbf{b} + \mathbf{b}\times\mathbf{r}$ where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\mathbf{b}$ is a constant vector.

In accordance with *(1.59)* and *(1.60)*, construct the vectors

$$\begin{aligned} \mathbf{u} &= \mathbf{f}(\hat{\mathbf{i}}) = \hat{\mathbf{i}} + \hat{\mathbf{i}}\times\mathbf{r} = \hat{\mathbf{i}} - z\hat{\mathbf{j}} + y\hat{\mathbf{k}} \\ \mathbf{v} &= \mathbf{f}(\hat{\mathbf{j}}) = \hat{\mathbf{j}} + \hat{\mathbf{j}}\times\mathbf{r} = z\hat{\mathbf{i}} + \hat{\mathbf{j}} - x\hat{\mathbf{k}} \\ \mathbf{w} &= \mathbf{f}(\hat{\mathbf{k}}) = \hat{\mathbf{k}} + \hat{\mathbf{k}}\times\mathbf{r} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + \hat{\mathbf{k}} \end{aligned}$$

Then $\quad \mathbf{D} = \mathbf{u}\hat{\mathbf{i}} + \mathbf{v}\hat{\mathbf{j}} + \mathbf{w}\hat{\mathbf{k}} = (\hat{\mathbf{i}} - z\hat{\mathbf{j}} + y\hat{\mathbf{k}})\hat{\mathbf{i}} + (z\hat{\mathbf{i}} + \hat{\mathbf{j}} - x\hat{\mathbf{k}})\hat{\mathbf{j}} + (-y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + \hat{\mathbf{k}})\hat{\mathbf{k}}$

and $\quad \mathbf{a} = \mathbf{D}\cdot\mathbf{b} = (b_x + b_y z - b_z y)\hat{\mathbf{i}} + (-b_x z + b_y + b_z x)\hat{\mathbf{j}} + (b_x y - b_y x + b_z)\hat{\mathbf{k}}$

As a check the same result may be obtained by direct expansion of the vector function,

$$\mathbf{a} = \mathbf{b} + \mathbf{b}\times\mathbf{r} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}} + (b_y z - b_z y)\hat{\mathbf{i}} + (b_z x - b_x z)\hat{\mathbf{j}} + (b_x y - b_y x)\hat{\mathbf{k}}$$

1.18. Express the unit triad $\hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_r$ in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and confirm that the curvilinear triad is right-handed by showing that $\hat{\mathbf{e}}_\phi \times \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_r$.

By direct projection from Fig. 1-15,

$$\begin{aligned} \hat{\mathbf{e}}_\phi &= (\cos\phi\cos\theta)\hat{\mathbf{i}} + (\cos\phi\sin\theta)\hat{\mathbf{j}} - (\sin\phi)\hat{\mathbf{k}} \\ \hat{\mathbf{e}}_\theta &= (-\sin\theta)\hat{\mathbf{i}} + (\cos\theta)\hat{\mathbf{j}} \\ \hat{\mathbf{e}}_r &= (\sin\phi\cos\theta)\hat{\mathbf{i}} + (\sin\phi\sin\theta)\hat{\mathbf{j}} + (\cos\phi)\hat{\mathbf{k}} \end{aligned}$$

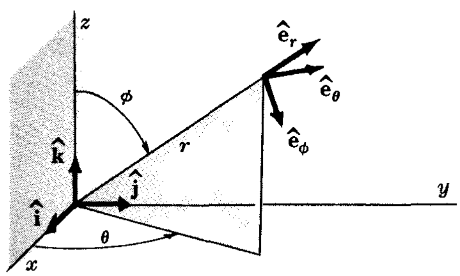


Fig. 1-15

and so

$$\begin{aligned} \hat{\mathbf{e}}_\phi \times \hat{\mathbf{e}}_\theta &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} \\ &= (\sin\phi\cos\theta)\hat{\mathbf{i}} + (\sin\phi\sin\theta)\hat{\mathbf{j}} + [(\cos^2\theta + \sin^2\theta)\cos\phi]\hat{\mathbf{k}} = \hat{\mathbf{e}}_r \end{aligned}$$

1.19. Resolve the dyadic $\mathbf{D} = 3\hat{\mathbf{i}}\hat{\mathbf{i}} + 4\hat{\mathbf{i}}\hat{\mathbf{k}} + 6\hat{\mathbf{j}}\hat{\mathbf{i}} + 7\hat{\mathbf{j}}\hat{\mathbf{j}} + 10\hat{\mathbf{k}}\hat{\mathbf{i}} + 2\hat{\mathbf{k}}\hat{\mathbf{j}}$ into its symmetric and antisymmetric parts.

Let $\mathbf{D} = \mathbf{E} + \mathbf{F}$ where $\mathbf{E} = \mathbf{E}_c$ and $\mathbf{F} = -\mathbf{F}_c$. Then

$$\begin{aligned} \mathbf{E} = (1/2)(\mathbf{D} + \mathbf{D}_c) &= (1/2)(6\hat{\mathbf{i}}\hat{\mathbf{i}} + 4\hat{\mathbf{i}}\hat{\mathbf{k}} + 4\hat{\mathbf{k}}\hat{\mathbf{i}} + 6\hat{\mathbf{j}}\hat{\mathbf{i}} + 6\hat{\mathbf{i}}\hat{\mathbf{j}} + 14\hat{\mathbf{j}}\hat{\mathbf{j}} \\ &\qquad + 10\hat{\mathbf{k}}\hat{\mathbf{i}} + 10\hat{\mathbf{i}}\hat{\mathbf{k}} + 2\hat{\mathbf{k}}\hat{\mathbf{j}} + 2\hat{\mathbf{j}}\hat{\mathbf{k}}) \\ &= 3\hat{\mathbf{i}}\hat{\mathbf{i}} + 3\hat{\mathbf{i}}\hat{\mathbf{j}} + 7\hat{\mathbf{i}}\hat{\mathbf{k}} + 3\hat{\mathbf{j}}\hat{\mathbf{i}} + 7\hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{j}}\hat{\mathbf{k}} + 7\hat{\mathbf{k}}\hat{\mathbf{i}} + \hat{\mathbf{k}}\hat{\mathbf{j}} = \mathbf{E}_c \\ \mathbf{F} = (1/2)(\mathbf{D} - \mathbf{D}_c) &= (1/2)(4\hat{\mathbf{i}}\hat{\mathbf{k}} - 4\hat{\mathbf{k}}\hat{\mathbf{i}} + 6\hat{\mathbf{j}}\hat{\mathbf{i}} - 6\hat{\mathbf{i}}\hat{\mathbf{j}} + 10\hat{\mathbf{k}}\hat{\mathbf{i}} - 10\hat{\mathbf{i}}\hat{\mathbf{k}} + 2\hat{\mathbf{k}}\hat{\mathbf{j}} - 2\hat{\mathbf{j}}\hat{\mathbf{k}}) \\ &= -3\hat{\mathbf{i}}\hat{\mathbf{j}} - 3\hat{\mathbf{i}}\hat{\mathbf{k}} + 3\hat{\mathbf{j}}\hat{\mathbf{i}} - \hat{\mathbf{j}}\hat{\mathbf{k}} + 3\hat{\mathbf{k}}\hat{\mathbf{i}} + \hat{\mathbf{k}}\hat{\mathbf{j}} = -\mathbf{F}_c \end{aligned}$$

1.20. With respect to the set of base vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ (not necessarily unit vectors), the set $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ is said to be a reciprocal basis if $\mathbf{a}_i \cdot \mathbf{a}^j = \delta_{ij}$. Determine the necessary relationships for constructing the reciprocal base vectors and carry out the calculations for the basis

$$\mathbf{b}_1 = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}, \quad \mathbf{b}_2 = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \quad \mathbf{b}_3 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

By definition, $\mathbf{a}_1 \cdot \mathbf{a}^1 = 1$, $\mathbf{a}_2 \cdot \mathbf{a}^1 = 0$, $\mathbf{a}_3 \cdot \mathbf{a}^1 = 0$. Hence $\mathbf{a}^1$ is perpendicular to both $\mathbf{a}_2$ and $\mathbf{a}_3$. Therefore it is parallel to $\mathbf{a}_2 \times \mathbf{a}_3$, i.e. $\mathbf{a}^1 = \lambda(\mathbf{a}_2 \times \mathbf{a}_3)$. Since $\mathbf{a}_1 \cdot \mathbf{a}^1 = 1$, $\mathbf{a}_1 \cdot \lambda \mathbf{a}_2 \times \mathbf{a}_3 = 1$ and $\lambda = 1/(\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3) = 1/[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]$. Thus, in general,

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]}, \quad \mathbf{a}^2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]}, \quad \mathbf{a}^3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]}$$

For the basis $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, $1/\lambda = \mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3 = 12$ and so

$$\begin{aligned} \mathbf{b}^1 &= (\mathbf{b}_2 \times \mathbf{b}_3)/12 = (\hat{\mathbf{j}} - \hat{\mathbf{k}})/4 \\ \mathbf{b}^2 &= (\mathbf{b}_3 \times \mathbf{b}_1)/12 = -\hat{\mathbf{i}}/3 + \hat{\mathbf{j}}/4 + \hat{\mathbf{k}}/12 \\ \mathbf{b}^3 &= (\mathbf{b}_1 \times \mathbf{b}_2)/12 = 2\hat{\mathbf{i}}/3 - \hat{\mathbf{j}}/2 + 5\hat{\mathbf{k}}/6 \end{aligned}$$

INDICIAL NOTATION — CARTESIAN TENSORS (Sec. 1.9-1.16)

1.21. For a range of three on the indices, give the meaning of the following Cartesian tensor symbols: A_{ii}, B_{ijj}, R_{ij}, $a_i T_{ij}$, $a_i b_j S_{ij}$.

A_{ii} represents the single sum $A_{ii} = A_{11} + A_{22} + A_{33}$.

B_{ijj} represents three sums: (1) For $i = 1$, $B_{111} + B_{122} + B_{133}$.
(2) For $i = 2$, $B_{211} + B_{222} + B_{233}$.
(3) For $i = 3$, $B_{311} + B_{322} + B_{333}$.

R_{ij} represents the nine components $R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}, R_{33}$.

$a_i T_{ij}$ represents three sums: (1) For $j = 1$, $a_1 T_{11} + a_2 T_{21} + a_3 T_{31}$.
(2) For $j = 2$, $a_1 T_{12} + a_2 T_{22} + a_3 T_{32}$.
(3) For $j = 3$, $a_1 T_{13} + a_2 T_{23} + a_3 T_{33}$.

$a_i b_j S_{ij}$ represents a single sum of nine terms. Summing first on i, $a_i b_j S_{ij} = a_1 b_j S_{1j} + a_2 b_j S_{2j} + a_3 b_j S_{3j}$. Now summing each of these three terms on j,

$$\begin{aligned} a_i b_j S_{ij} = {} & a_1 b_1 S_{11} + a_1 b_2 S_{12} + a_1 b_3 S_{13} + a_2 b_1 S_{21} + a_2 b_2 S_{22} \\ & + a_2 b_3 S_{23} + a_3 b_1 S_{31} + a_3 b_2 S_{32} + a_3 b_3 S_{33} \end{aligned}$$

1.22. Evaluate the following expressions involving the Kronecker delta δ_{ij} for a range of three on the indices.

(a) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(b) $\delta_{ij}\delta_{ij} = \delta_{1j}\delta_{1j} + \delta_{2j}\delta_{2j} + \delta_{3j}\delta_{3j} = 3$

(c) $\delta_{ij}\delta_{ik}\delta_{jk} = \delta_{1j}\delta_{1k}\delta_{jk} + \delta_{2j}\delta_{2k}\delta_{jk} + \delta_{3j}\delta_{3k}\delta_{jk} = 3$

(d) $\delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k} = \delta_{ik}$

(e) $\delta_{ij}A_{ik} = \delta_{1j}A_{1k} + \delta_{2j}A_{2k} + \delta_{3j}A_{3k} = A_{jk}$

1.23. For the permutation symbol ϵ_{ijk} show by direct expansion that (a) $\epsilon_{ijk}\epsilon_{kij} = 6$, (b) $\epsilon_{ijk}a_j a_k = 0$.

(a) First sum on i, $\epsilon_{ijk}\epsilon_{kij} = \epsilon_{1jk}\epsilon_{k1j} + \epsilon_{2jk}\epsilon_{k2j} + \epsilon_{3jk}\epsilon_{k3j}$

Next sum on j. The nonzero terms are

$$\epsilon_{ijk}\epsilon_{kij} = \epsilon_{12k}\epsilon_{k12} + \epsilon_{13k}\epsilon_{k13} + \epsilon_{21k}\epsilon_{k21} + \epsilon_{23k}\epsilon_{k23} + \epsilon_{31k}\epsilon_{k31} + \epsilon_{32k}\epsilon_{k32}$$

Finally summing on k, the nonzero terms are

$$\begin{aligned}\epsilon_{ijk}\epsilon_{kij} &= \epsilon_{123}\epsilon_{312} + \epsilon_{132}\epsilon_{213} + \epsilon_{213}\epsilon_{321} + \epsilon_{231}\epsilon_{123} + \epsilon_{312}\epsilon_{231} + \epsilon_{321}\epsilon_{132} \\ &= (1)(1) + (-1)(-1) + (-1)(-1) + (1)(1) + (1)(1) + (-1)(-1) = 6\end{aligned}$$

(*b*) Summing on j and k in turn,

$$\begin{aligned}\epsilon_{ijk}a_j a_k &= \epsilon_{i1k}a_1 a_k + \epsilon_{i2k}a_2 a_k + \epsilon_{i3k}a_3 a_k \\ &= \epsilon_{i12}a_1 a_2 + \epsilon_{i13}a_1 a_3 + \epsilon_{i21}a_2 a_1 + \epsilon_{i23}a_2 a_3 + \epsilon_{i31}a_3 a_1 + \epsilon_{i32}a_3 a_2\end{aligned}$$

From this expression,

when $i = 1$, $\epsilon_{1jk}a_j a_k = a_2 a_3 - a_3 a_2 = 0$

when $i = 2$, $\epsilon_{2jk}a_j a_k = a_1 a_3 - a_3 a_1 = 0$

when $i = 3$, $\epsilon_{3jk}a_j a_k = a_1 a_2 - a_2 a_1 = 0$

Note that $\epsilon_{ijk}a_j a_k$ is the indicial form of the vector a crossed into itself, and so $\mathbf{a} \times \mathbf{a} = 0$.

1.24. Determine the component f_2 for the vector expressions given below.

(*a*) $f_i = \epsilon_{ijk}T_{jk}$

$f_2 = \epsilon_{2jk}T_{jk} = \epsilon_{213}T_{13} + \epsilon_{231}T_{31} = -T_{13} + T_{31}$

(*b*) $f_i = c_{i,j}b_j - c_{j,i}b_j$

$$\begin{aligned}f_2 &= c_{2,1}b_1 + c_{2,2}b_2 + c_{2,3}b_3 - c_{1,2}b_1 - c_{2,2}b_2 - c_{3,2}b_3 \\ &= (c_{2,1} - c_{1,2})b_1 + (c_{2,3} - c_{3,2})b_3\end{aligned}$$

(*c*) $f_i = B_{ij}f_j^*$

$f_2 = B_{21}f_1^* + B_{22}f_2^* + B_{23}f_3^*$

1.25. Expand and simplify where possible the expression $D_{ij}x_ix_j$ for (*a*) $D_{ij} = D_{ji}$, (*b*) $D_{ij} = -D_{ji}$.

Expanding,
$$\begin{aligned}D_{ij}x_ix_j &= D_{1j}x_1x_j + D_{2j}x_2x_j + D_{3j}x_3x_j \\ &= D_{11}x_1x_1 + D_{12}x_1x_2 + D_{13}x_1x_3 + D_{21}x_2x_1 + D_{22}x_2x_2 \\ &\quad + D_{23}x_2x_3 + D_{31}x_3x_1 + D_{32}x_3x_2 + D_{33}x_3x_3\end{aligned}$$

(*a*) $D_{ij}x_ix_j = D_{11}(x_1)^2 + D_{22}(x_2)^2 + D_{33}(x_3)^2 + 2D_{12}x_1x_2 + 2D_{23}x_2x_3 + 2D_{13}x_1x_3$

(*b*) $D_{ij}x_ix_j = 0$ since $D_{11} = -D_{11}$, $D_{12} = -D_{21}$, etc.

1.26. Show that $\epsilon_{ijk}\epsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$ for (*a*) $i = 1$, $j = q = 2$, $p = 3$ and for (*b*) $i = q = 1$, $j = p = 2$. (It is shown in Problem 1.59 that this identity holds for every choice of indices.)

(*a*) Introduce $i = 1$, $j = 2$, $p = 3$, $q = 2$ and note that since k is a summed index it takes on all values. Then

$$\epsilon_{ijk}\epsilon_{kpq} = \epsilon_{12k}\epsilon_{k32} = \epsilon_{121}\epsilon_{132} + \epsilon_{122}\epsilon_{232} + \epsilon_{123}\epsilon_{332} = 0$$

and

$$\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} = \delta_{13}\delta_{22} - \delta_{12}\delta_{23} = 0$$

(*b*) Introduce $i = 1$, $j = 2$, $p = 2$, $q = 1$. Then $\epsilon_{ijk}\epsilon_{kpq} = \epsilon_{123}\epsilon_{321} = -1$ and $\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} = \delta_{12}\delta_{21} - \delta_{11}\delta_{22} = -1$.

1.27. Show that the tensor $B_{ik} = \epsilon_{ijk}a_j$ is skew-symmetric.

Since by definition of ϵ_{ijk} an interchange of two indices causes a sign change,

$$B_{ik} = \epsilon_{ijk}a_j = -(\epsilon_{kji}a_j) = -(B_{ki}) = -B_{ki}$$

1.28. If B_{ij} is a skew-symmetric Cartesian tensor for which the vector $b_i = (\frac{1}{2})\epsilon_{ijk}B_{jk}$, show that $B_{pq} = \epsilon_{pqi}b_i$.

Multiply the given equation by ϵ_{pqi} and use the identity given in Problem 1.26.

$$\epsilon_{pqi}b_i = \tfrac{1}{2}\epsilon_{pqi}\epsilon_{ijk}B_{jk} = \tfrac{1}{2}(\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})B_{jk} = \tfrac{1}{2}(B_{pq} - B_{qp}) = \tfrac{1}{2}(B_{pq} + B_{pq}) = B_{pq}$$

1.29. Determine directly the components of the metric tensor for spherical polar coordinates as shown in Fig. 1-7(*b*).

Write (*1.87*) as $g_{pq} = \dfrac{\partial x_i}{\partial \theta_p}\dfrac{\partial x_i}{\partial \theta_q}$ and label the coordinates as shown in Fig. 1-16 ($r = \theta_1$, $\phi = \theta_2$, $\theta = \theta_3$). Then

$$x_1 = \theta_1 \sin\theta_2 \cos\theta_3$$

$$x_2 = \theta_1 \sin\theta_2 \sin\theta_3$$

$$x_3 = \theta_1 \cos\theta_2$$

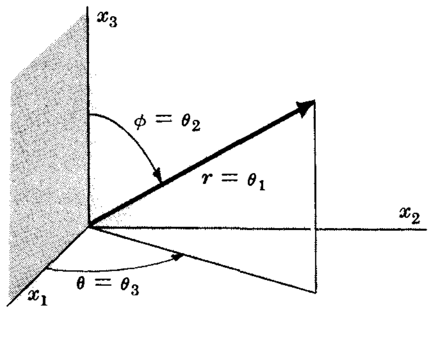


Fig. 1-16

Hence

$$\frac{\partial x_1}{\partial \theta_1} = \sin\theta_2\cos\theta_3 \qquad \frac{\partial x_1}{\partial \theta_2} = \theta_1\cos\theta_2\cos\theta_3 \qquad \frac{\partial x_1}{\partial \theta_3} = -\theta_1\sin\theta_2\sin\theta_3$$

$$\frac{\partial x_2}{\partial \theta_1} = \sin\theta_2\sin\theta_3 \qquad \frac{\partial x_2}{\partial \theta_2} = \theta_1\cos\theta_2\sin\theta_3 \qquad \frac{\partial x_2}{\partial \theta_3} = \theta_1\sin\theta_2\cos\theta_3$$

$$\frac{\partial x_3}{\partial \theta_1} = \cos\theta_2 \qquad \frac{\partial x_3}{\partial \theta_2} = -\theta_1\sin\theta_2 \qquad \frac{\partial x_3}{\partial \theta_3} = 0$$

from which $$g_{11} = \frac{\partial x_i}{\partial \theta_1}\frac{\partial x_i}{\partial \theta_1} = \sin^2\theta_2\cos^2\theta_3 + \sin^2\theta_2\sin^2\theta_3 + \cos^2\theta_2 = 1$$

$$g_{22} = \frac{\partial x_i}{\partial \theta_2}\frac{\partial x_i}{\partial \theta_2} = \theta_1^2\cos^2\theta_2\cos^2\theta_3 + \theta_1^2\cos^2\theta_2\sin^2\theta_3 + \theta_1^2\sin^2\theta_2 = \theta_1^2$$

$$g_{33} = \frac{\partial x_i}{\partial \theta_3}\frac{\partial x_i}{\partial \theta_3} = \theta_1^2\sin^2\theta_2\sin^2\theta_3 + \theta_1^2\sin^2\theta_2\cos^2\theta_3 = \theta_1^2\sin^2\theta_2$$

Also, $g_{pq} = 0$ for $p \neq q$. For example,

$$g_{12} = \frac{\partial x_i}{\partial \theta_1}\frac{\partial x_i}{\partial \theta_2} = (\sin\theta_2\cos\theta_3)(\theta_1\cos\theta_2\cos\theta_3) + (\sin\theta_2\sin\theta_3)(\theta_1\cos\theta_2\sin\theta_3) - (\cos\theta_2)(\theta_1\sin\theta_2) = 0$$

Thus for spherical coordinates, $(ds)^2 = (d\theta_1)^2 + (\theta_1)^2(d\theta_2)^2 + (\theta_1\sin\theta_2)^2(d\theta_3)^2$.

1.30. Show that the length of the line element ds resulting from the curvilinear coordinate increment $d\theta_i$ is given by $ds = \sqrt{g_{ii}}\,d\theta_i$ (no sum). Apply this result to the spherical coordinate system of Problem 1.29.

Write (*1.86*) as $(ds)^2 = g_{pq}\,d\theta_p\,d\theta_q$. Thus for the line element $(d\theta_1, 0, 0)$, it follows that $(ds)^2 = g_{11}(d\theta_1)^2$ and $ds = \sqrt{g_{11}}\,d\theta_1$. Similarly for $(0, d\theta_2, 0)$, $ds = \sqrt{g_{22}}\,d\theta_2$; and for $(0, 0, d\theta_3)$, $ds = \sqrt{g_{33}}\,d\theta_3$. Therefore (Fig. 1-17),

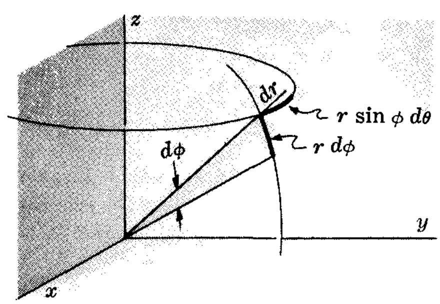


Fig. 1-17

(1) For $(d\theta_1, 0, 0)$, $\quad ds = d\theta_1 = dr$

(2) For $(0, d\theta_2, 0)$, $\quad ds = \theta_1\, d\theta_2 = r\, d\phi$

(3) For $(0, 0, d\theta_3)$, $\quad ds = \theta_1 \sin\theta_2\, d\theta_3 = r \sin\phi\, d\theta$

1.31. If the angle between the line elements represented by $(d\theta_1, 0, 0)$ and $(0, d\theta_2, 0)$ is denoted by β_{12}, show that $\cos\beta_{12} = \dfrac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$.

Let $ds_1 = \sqrt{g_{11}}\, d\theta_1$ be the length of line element represented by $(d\theta_1, 0, 0)$ and $ds_2 = \sqrt{g_{22}}\, d\theta_2$ be that of $(0, d\theta_2, 0)$. Write *(1.85)* as $dx_i = \dfrac{\partial x_i}{\partial \theta_k}\, d\theta_k$, and since $(ds)^2 = \cos\beta_{12}\, ds_1\, ds_2$,

$$\begin{aligned}(ds)^2 &= dx_i\, dx_i = \delta_{ij}\, dx_i\, dx_j \\ &= \frac{\partial x_1}{\partial\theta_1}\frac{\partial x_1}{\partial\theta_2} d\theta_1\, d\theta_2 + \frac{\partial x_2}{\partial\theta_1}\frac{\partial x_2}{\partial\theta_2} d\theta_1\, d\theta_2 + \frac{\partial x_3}{\partial\theta_1}\frac{\partial x_3}{\partial\theta_2} d\theta_1\, d\theta_2 = g_{12}\, d\theta_1\, d\theta_2\end{aligned}$$

Hence using the result of Problem 1.30, $\cos\beta_{12} = g_{12}\dfrac{d\theta_1}{ds_1}\dfrac{d\theta_2}{ds_2} = \dfrac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$.

1.32. A primed set of Cartesian axes $Ox_1'x_2'x_3'$ is obtained by a rotation through an angle θ about the x_3 axis. Determine the transformation coefficients a_{ij} relating the axes, and give the primed components of the vector $\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$.

From the definition (see Section 1.13) $a_{ij} = \cos(x_i, x_j)$ and Fig. 1-18, the table of direction cosines is

	x_1	x_2	x_3
x_1'	$\cos\theta$	$\sin\theta$	0
x_2'	$-\sin\theta$	$\cos\theta$	0
x_3'	0	0	1

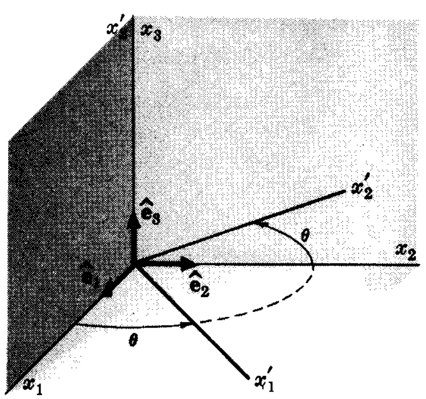


Fig. 1-18

Thus the transformation tensor is

$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By the transformation law for vectors *(1.94)*,

$$\begin{aligned} v_1' &= a_{1j}v_j = v_1\cos\theta + v_2\sin\theta \\ v_2' &= a_{2j}v_j = -v_1\sin\theta + v_2\cos\theta \\ v_3' &= a_{3j}v_j = v_3 \end{aligned}$$

1.33. The table of direction cosines relating two sets of rectangular Cartesian axes is partially given as shown on the right. Determine the entries for the bottom row of the table so that $Ox_1'x_2'x_3'$ is a right-handed system.

	x_1	x_2	x_3
x_1'	3/5	−4/5	0
x_2'	0	0	1
x_3'			

The unit vector $\hat{\mathbf{e}}_1'$ along the x_1' axis is given by the first row of the table as $\hat{\mathbf{e}}_1' = (3/5)\hat{\mathbf{e}}_1 - (4/5)\hat{\mathbf{e}}_2$. Also from the table $\hat{\mathbf{e}}_2' = \hat{\mathbf{e}}_3$. For a right-handed primed system $\hat{\mathbf{e}}_3' = \hat{\mathbf{e}}_1' \times \hat{\mathbf{e}}_2'$, or $\hat{\mathbf{e}}_3' = [(3/5)\hat{\mathbf{e}}_1 - (4/5)\hat{\mathbf{e}}_2] \times \hat{\mathbf{e}}_3 = (-3/5)\hat{\mathbf{e}}_2 - (4/5)\hat{\mathbf{e}}_1$ and the third row is

x_3'	$-4/5$	$-3/5$	0

.

1.34. Let the *angles* between the primed and unprimed coordinate directions be given by the table shown on the right. Determine the transformation coefficients a_{ij} and show that the orthogonality conditions are satisfied.

	x_1	x_2	x_3
x_1'	135°	60°	120°
x_2'	90°	45°	45°
x_3'	45°	60°	120°

The coefficients a_{ij} are direction cosines and may be calculated directly from the table. Thus

$$a_{ij} = \begin{pmatrix} -1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & -1/2 \end{pmatrix}$$

The orthogonality conditions $a_{ij}a_{ik} = \delta_{jk}$ require:

1. For $j = k = 1$ that $a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31} = 1$ which is seen to be the sum of squares of the elements in the first column.
2. For $j = 2,\ k = 3$ that $a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0$ which is seen to be the sum of products of corresponding elements of the second and third columns.
3. Any two columns "multiplied together element by element and summed" to be zero. The sum of squares of elements of any column to be unity.

For orthogonality conditions in the form $a_{ji}a_{ki} = \delta_{jk}$, the rows are multiplied together instead of the columns. All of these conditions are satisfied by the above solution.

1.35. Show that the sum $\lambda A_{ij} + \mu B_{ij}$ represents the components of a second-order tensor if A_{ij} and B_{ij} are known second-order tensors.

By (*1.103*) and the statement of the problem, $A_{ij} = a_{pi}a_{qj}A'_{pq}$ and $B_{ij} = a_{pi}a_{qj}B'_{pq}$. Hence

$$\lambda A_{ij} + \mu B_{ij} = \lambda(a_{pi}a_{qj}A'_{pq}) + \mu(a_{pi}a_{qj}B'_{pq}) = a_{pi}a_{qj}(\lambda A'_{pq} + \mu B'_{pq})$$

which demonstrates that the sum transforms as a second-order Cartesian tensor.

1.36. Show that $(P_{ijk} + P_{jki} + P_{jik})\,x_ix_jx_k = 3P_{ijk}\,x_ix_jx_k$.

Since all indices are dummy indices and the order of the variables x_i is unimportant, each term of the sum is equivalent to the others. This may be readily shown by introducing new dummy variables. Thus replacing i, j, k in the second and third terms by p, q, r, the sum becomes

$$P_{ijk}x_ix_jx_k + P_{qrp}x_px_qx_r + P_{qpr}x_px_qx_r$$

Now change dummy indices in these same terms again so that the form is

$$P_{ijk}\,x_ix_jx_k + P_{ijk}\,x_kx_ix_j + P_{ijk}\,x_jx_ix_k = 3P_{ijk}\,x_ix_jx_k$$

1.37. If B_{ij} is skew-symmetric and A_{ij} is symmetric, show that $A_{ij}B_{ij} = 0$.

Since $A_{ij} = A_{ji}$ and $B_{ij} = -B_{ji}$, $A_{ij}B_{ij} = -A_{ji}B_{ji}$ or $A_{ij}B_{ij} + A_{ji}B_{ji} = A_{ij}B_{ij} + A_{pq}B_{pq} = 0$. Since all indices are dummy indices, $A_{pq}B_{pq} = A_{ij}B_{ij}$ and so $2A_{ij}B_{ij} = 0$, or $A_{ij}B_{ij} = 0$.

1.38. Show that the quadratic form $D_{ij}x_ix_j$ is unchanged if D_{ij} is replaced by its symmetric part $D_{(ij)}$.

Resolving D_{ij} into its symmetric and anti-symmetric parts,

$$D_{ij} = D_{(ij)} + D_{[ij]} = \tfrac{1}{2}(D_{ij} + D_{ji}) + \tfrac{1}{2}(D_{ij} - D_{ji})$$

Then $$D_{(ij)}x_ix_j = \tfrac{1}{2}(D_{ij} + D_{ji})x_ix_j = \tfrac{1}{2}(D_{ij}x_ix_j + D_{pq}x_qx_p) = D_{ij}x_ix_j$$

1.39. Use indicial notation to prove the vector identities

$$(1)\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \qquad (2)\ \mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0$$

(1) Let $\mathbf{v} = \mathbf{b} \times \mathbf{c}$. Then $v_i = \epsilon_{ijk}b_jc_k$; and if $\mathbf{a} \times \mathbf{v} = \mathbf{w}$, then

$$\begin{aligned} w_p &= \epsilon_{pqi}a_q\epsilon_{ijk}b_jc_k \\ &= (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})a_qb_jc_k \qquad \text{(see Problem 1.26)} \\ &= a_qb_pc_q - a_qb_qc_p \\ &= (a_qc_q)b_p - (a_qb_q)c_p \end{aligned}$$

Transcribing this expression into symbolic notation,

$$\mathbf{w} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

(2) Let $\mathbf{a} \times \mathbf{b} = \mathbf{v}$. Thus $v_i = \epsilon_{ijk}a_jb_k$; and if $\lambda = \mathbf{v} \cdot \mathbf{a}$, then $\lambda = \epsilon_{ijk}(a_ia_jb_k)$. But ϵ_{ijk} is skew-symmetric in i and j, while $(a_ia_jb_k)$ is symmetric in i and j. Hence the product $\epsilon_{ijk}a_ia_jb_k$ vanishes as may also be shown by direct expansion.

$$\begin{aligned} \lambda &= \epsilon_{ij1}a_ia_jb_1 + \epsilon_{ij2}a_ia_jb_2 + \epsilon_{ij3}a_ia_jb_3 \\ &= (\epsilon_{321}a_3a_2 + \epsilon_{231}a_2a_3)b_1 + \cdots \\ &= (-a_2a_3 + a_2a_3)b_1 + (0)b_2 + (0)b_3 = 0 \end{aligned}$$

1.40. Show that the determinant

$$\det A_{ij} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

may be expressed in the form $\epsilon_{ijk}A_{1i}A_{2j}A_{3k}$.

From *(1.52)* and *(1.109)* the box product [abc] may be written

$$\lambda = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = [\mathbf{abc}] = \epsilon_{ijk}a_ib_jc_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If now the substitutions $a_i = A_{1i}$, $b_i = A_{2i}$ and $c_i = A_{3i}$ are introduced,

$$\lambda = \epsilon_{ijk}a_ib_jc_k = \epsilon_{ijk}A_{1i}A_{2j}A_{3k}$$

This result may also be obtained by direct expansion of the determinant. An equivalent expression for the determinant is $\epsilon_{ijk}A_{i1}A_{j2}A_{k3}$.

1.41. If the vector v_i is given in terms of base vectors **a, b, c** by $v_i = \alpha a_i + \beta b_i + \gamma c_i$, show that $\alpha = \dfrac{\epsilon_{ijk}v_ib_jc_k}{\epsilon_{pqr}a_pb_qc_r}$.

$$\begin{aligned} v_1 &= \alpha a_1 + \beta b_1 + \gamma c_1 \\ v_2 &= \alpha a_2 + \beta b_2 + \gamma c_2 \\ v_3 &= \alpha a_3 + \beta b_3 + \gamma c_3 \end{aligned}$$

By Cramer's rule, $\alpha = \dfrac{\begin{vmatrix} v_1 & b_1 & c_1 \\ v_2 & b_2 & c_2 \\ v_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$ and by (*1.52*) and (*1.109*), $\alpha = \dfrac{\epsilon_{ijk} v_i b_j c_k}{\epsilon_{pqr} a_p b_q c_r}$.

Likewise $\beta = \dfrac{\epsilon_{ijk} a_i v_j c_k}{\epsilon_{pqr} a_p b_q c_r}$, $\gamma = \dfrac{\epsilon_{ijk} a_i b_j v_k}{\epsilon_{pqr} a_p b_q c_r}$.

MATRICES AND MATRIX METHODS (Sec. 1.17-1.20)

1.42. For the vectors $\mathbf{a} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$ and the dyadic $\mathbf{D} = 3\hat{\mathbf{i}}\hat{\mathbf{i}} + 2\hat{\mathbf{i}}\hat{\mathbf{k}} - 4\hat{\mathbf{j}}\hat{\mathbf{j}} - 5\hat{\mathbf{k}}\hat{\mathbf{j}}$, compute by matrix multiplication the products $\mathbf{a}\cdot\mathbf{D}$, $\mathbf{D}\cdot\mathbf{b}$ and $\mathbf{a}\cdot\mathbf{D}\cdot\mathbf{b}$.

Let $\mathbf{a}\cdot\mathbf{D} = \mathbf{v}$; then $[v_1, v_2, v_3] = [3, 0, 4]\begin{bmatrix} 3 & 0 & 2 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{bmatrix} = [9, -20, 6]$.

Let $\mathbf{D}\cdot\mathbf{b} = \mathbf{w}$; then $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \\ -10 \end{bmatrix}$.

Let $\mathbf{a}\cdot\mathbf{D}\cdot\mathbf{b} = \mathbf{v}\cdot\mathbf{b} = \lambda$; then $[\lambda] = [9, -20, 6]\begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix} = [-76]$.

1.43. Determine the principal directions and principal values of the second-order Cartesian tensor $\mathbf{T}$ whose matrix representation is

$$[T_{ij}] = \begin{vmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

From (*1.132*), for principal values λ,

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)[(3-\lambda)^2 - 1] = 0$$

which results in the cubic equation $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = (\lambda-1)(\lambda-2)(\lambda-4) = 0$ whose principal values are $\lambda_{(1)} = 1$, $\lambda_{(2)} = 2$, $\lambda_{(3)} = 4$.

Next let $n_i^{(1)}$ be the components of the unit normal in the principal direction associated with $\lambda_{(1)} = 1$. Then the first two equations of (*1.131*) give $2n_1^{(1)} - n_2^{(1)} = 0$ and $-n_1^{(1)} + 2n_2^{(1)} = 0$, from which $n_1^{(1)} = n_2^{(1)} = 0$; and from $n_i n_i = 1$, $n_3^{(1)} = \pm 1$.

For $\lambda_{(2)} = 2$, (*1.131*) yields $n_1^{(2)} - n_2^{(2)} = 0$, $-n_1^{(2)} + n_2^{(2)} = 0$, and $-n_3^{(2)} = 0$. Thus $n_1^{(2)} = n_2^{(2)} = \pm 1/\sqrt{2}$, since $n_i n_i = 1$ and $n_3^{(2)} = 0$.

For $\lambda_{(3)} = 4$, (*1.131*) yields $-n_1^{(3)} - n_2^{(3)} = 0$, $-n_1^{(3)} - n_2^{(3)} = 0$, and $3n_3^{(3)} = 0$. Thus $n_3^{(3)} = 0$, $n_1^{(3)} = -n_2^{(3)} = \mp 1/\sqrt{2}$.

The principal axes x_i^* may be referred to the original axes x_i through the table of direction cosines

	x_1	x_2	x_3
x_1^*	0	0	± 1
x_2^*	$\pm 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0
x_3^*	$\mp 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0

from which the transformation matrix (tensor) may be written:

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm 1/\sqrt{2} & \pm 1/\sqrt{2} & 0 \\ \mp 1/\sqrt{2} & \pm 1/\sqrt{2} & 0 \end{bmatrix} \quad \text{or} \quad a_{ij} = \begin{pmatrix} 0 & 0 & \pm 1 \\ \pm 1/\sqrt{2} & \pm 1/\sqrt{2} & 0 \\ \mp 1/\sqrt{2} & \pm 1/\sqrt{2} & 0 \end{pmatrix}$$

1.44. Show that the principal axes determined in Problem 1.43 form a right-handed set of orthogonal axes.

Orthogonality requires that the conditions $a_{ij}a_{ik} = \delta_{jk}$ be satisfied. Since the condition $n_in_i = 1$ was used in determining the a_{ij}, orthogonality is automatically satisfied for $j = k$. Multiplying the corresponding elements of any row (column) by those of any other row (column) and adding the products demonstrates that the conditions for $j \neq k$ are satisfied by the solution in Problem 1.43.

Finally for the system to be right-handed, $\hat{\mathbf{n}}^{(2)} \times \hat{\mathbf{n}}^{(3)} = \hat{\mathbf{n}}^{(1)}$. Thus

$$\begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{vmatrix} = (\tfrac{1}{2} + \tfrac{1}{2})\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$$

As indicated by the plus and minus values of a_{ij} in Problem 1.43, there are two sets of principal axes, x_i^* and x_i^{**}. As shown by the sketch both sets are along the principal directions with x_i^* being a right-handed system, x_i^{**} a left-handed system.

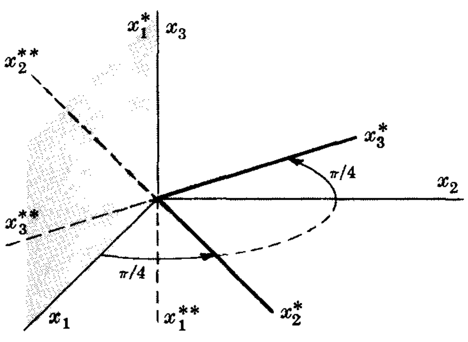


Fig. 1-19

1.45. Show that the matrix of the tensor T_{ij} of Problem 1.43 may be put into diagonal (principal) form by the transformation law $T_{ij}^* = a_{ip}a_{jq}T_{pq}$, (or in matrix symbols $\mathcal{T}^* = \mathcal{A}\mathcal{T}\mathcal{A}^T$).

$$[T_{ij}^*] = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \sqrt{2} & \sqrt{2} & 0 \\ -2\sqrt{2} & 2\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

1.46. Prove that if the principal values $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}$ of a symmetric second-order tensor are all distinct, the principal directions are mutually orthogonal.

The proof is made for $\lambda_{(2)}$ and $\lambda_{(3)}$. For each of these (*1.129*) is satisfied, so that $T_{ij}n_j^{(2)} = \lambda_{(2)}n_i^{(2)}$ and $T_{ij}n_j^{(3)} = \lambda_{(3)}n_i^{(3)}$. Multiplying the first of these equations by $n_i^{(3)}$ and the second by $n_i^{(2)}$,

$$T_{ij}n_j^{(2)}n_i^{(3)} = \lambda_{(2)}n_i^{(2)}n_i^{(3)}$$

$$T_{ij}n_j^{(3)}n_i^{(2)} = \lambda_{(3)}n_i^{(3)}n_i^{(2)}$$

Since T_{ij} is symmetric, the dummy indices i and j may be interchanged on the left-hand side of the second of these equations and that equation subtracted from the first to yield

$$(\lambda_{(2)} - \lambda_{(3)})n_i^{(2)}n_i^{(3)} = 0$$

Since $\lambda_{(2)} \neq \lambda_{(3)}$, their difference is not zero. Hence $n_i^{(2)}n_i^{(3)} = 0$, the condition for the two directions to be perpendicular.

1.47. Compute the principal values of $(\mathbf{T})^2$ of Problem 1.43 and verify that its principal axes coincide with those of $\mathbf{T}$.

$$[T_{ij}]^2 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic equation for this matrix is

$$\begin{vmatrix} 10-\lambda & -6 & 0 \\ -6 & 10-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)[(10-\lambda)^2 - 36] = (1-\lambda)(\lambda-4)(\lambda-16) = 0$$

from which $\lambda_{(1)} = 1$, $\lambda_{(2)} = 4$, $\lambda_{(3)} = 16$. Substituting these into (*1.131*) and using the condition $n_in_i = 1$,

For $\lambda_{(1)} = 1$, $$\left.\begin{aligned} 9n_1^{(1)} - 6n_2^{(1)} &= 0 \\ -6n_1^{(1)} + 9n_2^{(1)} &= 0 \end{aligned}\right\} \quad \text{or} \quad n_1^{(1)} = n_2^{(1)} = 0,\; n_3^{(1)} = \pm 1$$

For $\lambda_{(2)} = 4$, $$\left.\begin{aligned} 6n_1^{(2)} - 6n_2^{(2)} &= 0 \\ -6n_1^{(2)} + 6n_2^{(2)} &= 0 \\ -3n_3^{(2)} &= 0 \end{aligned}\right\} \quad \text{or} \quad n_1^{(2)} = n_2^{(2)} = \pm 1\sqrt{2},\; n_3^{(2)} = 0$$

For $\lambda_{(3)} = 16$, $$\left.\begin{aligned} -6n_1^{(3)} - 6n_2^{(3)} &= 0 \\ -6n_1^{(3)} - 6n_2^{(3)} &= 0 \\ -15n_3^{(3)} &= 0 \end{aligned}\right\} \quad \text{or} \quad n_1^{(3)} = -n_2^{(3)} = \mp 1/\sqrt{2},\; n_3^{(3)} = 0$$

which are the same as the principal directions of $\mathbf{T}$.

1.48. Use the fact that $(\mathbf{T})^2$ has the same principal directions as the symmetrical tensor $\mathbf{T}$ to obtain $\sqrt{\mathbf{T}}$ when

$$\mathbf{T} = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

First, the principal values and principal directions of $\mathbf{T}$ are determined. Following the procedure of Problem 1.43, the diagonal form of $\mathbf{T}$ is given by

$$\mathbf{T}^* = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

with the transformation matrix being

$$[a_{ij}] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

Therefore $\sqrt{\mathbf{T}^*} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{6} \end{pmatrix}$ and using $[a_{ij}]$ to relate this to the original axes by the transformation $\sqrt{\mathbf{T}} = \mathbf{A}_c\sqrt{\mathbf{T}^*}\,\mathbf{A}$, the matrix equation is

$$[\sqrt{T_{ij}}] = \begin{bmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$= \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2}+4 & \sqrt{2}-2 & \sqrt{2}-2 \\ \sqrt{2}-2 & \sqrt{2}+\sqrt{6}+1 & \sqrt{2}-\sqrt{6}+1 \\ \sqrt{2}-2 & \sqrt{2}-\sqrt{6}+1 & \sqrt{2}+\sqrt{6}+1 \end{bmatrix} = .402\begin{bmatrix} 5.414 & -0.586 & -0.586 \\ -0.586 & 4.863 & -0.035 \\ -0.586 & -0.035 & 4.863 \end{bmatrix}$$

CARTESIAN TENSOR CALCULUS (Sec. 1.21-1.23)

1.49. For the function $\lambda = A_{ij}x_ix_j$ where A_{ij} is constant, show that $\partial\lambda/\partial x_i = (A_{ij}+A_{ji})x_j$ and $\partial^2\lambda/\partial x_i\partial x_j = A_{ij}+A_{ji}$. Simplify these derivatives for the case $A_{ij} = A_{ji}$.

Consider $\dfrac{\partial\lambda}{\partial x_k} = A_{ij}\dfrac{\partial x_i}{\partial x_k}x_j + A_{ij}x_i\dfrac{\partial x_j}{\partial x_k}$. Since $\dfrac{\partial x_i}{\partial x_k} \equiv \delta_{ik}$, it is seen that $\dfrac{\partial\lambda}{\partial x_k} = A_{kj}x_j + A_{ik}x_i = (A_{kj}+A_{jk})x_j$. Continuing the differentiation, $\dfrac{\partial^2\lambda}{\partial x_p\,\partial x_k} = (A_{kj}+A_{jk})\dfrac{\partial x_j}{\partial x_p} = A_{kp}+A_{pk}$. If $A_{ij} = A_{ji}$, $\dfrac{\partial\lambda}{\partial x_k} = 2A_{kj}x_j$ and $\dfrac{\partial^2\lambda}{\partial x_p\,\partial x_k} = 2A_{pk}$.

1.50. Use indicial notation to prove the vector identities (*a*) $\nabla\times\nabla\phi = 0$, (*b*) $\nabla\cdot\nabla\times\mathbf{a} = 0$.

(*a*) By (*1.147*), $\nabla\phi$ is written $\phi_{,i}$ and so $\mathbf{v} = \nabla\times\nabla\phi$ has components $v_i = \epsilon_{ijk}\partial_j\phi_{,k} = \epsilon_{ijk}\phi_{,kj}$. But ϵ_{ijk} is anti-symmetric in j and k, whereas $\phi_{,kj}$ is symmetric in j and k; hence the product $\epsilon_{ijk}\phi_{,kj}$ vanishes. The same result may be found by computing individually the components of $\mathbf{v}$. For example, by expansion $v_1 = \epsilon_{123}\phi_{,23} + \epsilon_{132}\phi_{,32} = (\phi_{,23}-\phi_{,32}) = 0$.

(*b*) $\nabla\cdot\nabla\times\mathbf{a} = \lambda = (\epsilon_{ijk}a_{k,j})_{,i} = \epsilon_{ijk}a_{k,ji} = 0$ since $a_{k,ij} = a_{k,ji}$ and $\epsilon_{ijk} = -\epsilon_{jik}$.

1.51. Determine the derivative of the function $\lambda = (x_1)^2 + 2x_1x_2 - (x_3)^2$ in the direction of the unit normal $\hat{\mathbf{n}} = (2/7)\hat{\mathbf{e}}_1 - (3/7)\hat{\mathbf{e}}_2 - (6/7)\hat{\mathbf{e}}_3$ or $\hat{\mathbf{n}} = (2\hat{\mathbf{e}}_1 - 3\hat{\mathbf{e}}_2 - 6\hat{\mathbf{e}}_3)/7$.

The required derivative is $\dfrac{\partial\lambda}{\partial n} = \nabla\lambda\cdot\hat{\mathbf{n}} = \lambda_{,i}n_i$. Thus

$$\frac{\partial\lambda}{\partial n} = (2x_1+2x_2)\tfrac{2}{7} - (2x_1)\tfrac{3}{7} + (2x_3)\tfrac{6}{7} = \tfrac{2}{7}(-x_1+2x_2+6x_3)$$

1.52. If A_{ij} is a second-order Cartesian tensor, show that its derivative with respect to x_k, namely $A_{ij,k}$, is a third-order Cartesian tensor.

For the Cartesian coordinate systems x_i and x'_i, $x_i = a_{ji}x'_j$ and $\partial x_i/\partial x'_j = a_{ji}$. Hence

$$A'_{ij,k} = \frac{\partial(A'_{ij})}{\partial x'_k} = \frac{\partial}{\partial x'_k}(a_{ip}a_{jq}A_{pq}) = a_{ip}a_{jq}\frac{\partial A_{pq}}{\partial x_m}\frac{\partial x_m}{\partial x'_k} = a_{ip}a_{jq}a_{km}A_{pq,m}$$

which is the transformation law for a third-order Cartesian tensor.

1.53. If $r^2 = x_ix_i$ and $f(r)$ is an arbitrary function of r, show that (*a*) $\nabla(f(r)) = f'(r)\mathbf{x}/r$, and (*b*) $\nabla^2(f(r)) = f''(r) + 2f'/r$, where primes denote derivatives with respect to r.

(*a*) The components of ∇f are simply $f_{,i}$. Thus $f_{,i} = \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial x_i}$; and since $\dfrac{\partial(r^2)}{\partial x_j} = 2r\dfrac{\partial r}{\partial x_j} = 2\delta_{ij}x_i$ it follows that $\dfrac{\partial r}{\partial x_j} = \dfrac{x_j}{r}$. Thus $f_{,i} = \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial x_i} = f'x_i/r$.

(*b*) $$\nabla^2 f = f_{,ii} = (f'x_i/r)_{,i} = f''\frac{x_ix_i}{r^2} + f'\left(\frac{3}{r} - \frac{x_ix_i}{r^3}\right) = f'' + \frac{2f'}{r}.$$

1.54. Use the divergence theorem of Gauss to show that $\int_S x_in_j\,dS = V\delta_{ij}$ where $n_j\,dS$ represents the surface element of S, the bounding surface of the volume V shown in Fig. 1-20. x_i is the position vector of $n_j\,dS$, and n_i its outward normal.

By (*1.157*),

$$\int_S x_in_j\,dS = \int_V x_{i,j}\,dV = \int_V \delta_{ij}\,dV = \delta_{ij}V$$

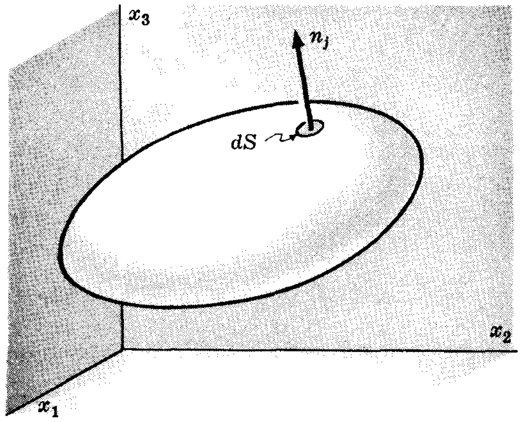


Fig. 1-20

1.55. If the vector $\mathbf{b} = \nabla \times \mathbf{v}$, show that $\int_S \lambda b_in_i\,dS = \int_V \lambda_{,i}b_i\,dV$ where $\lambda = \lambda(x_i)$ is a scalar function of the coordinates.

Since $\mathbf{b} = \nabla \times \mathbf{v}$, $b_i = \epsilon_{ijk}v_{k,j}$ and so

$$\begin{aligned}\int_S \lambda b_in_i\,dS &= \int_S \epsilon_{ijk}\lambda v_{k,j}n_i\,dS \\ &= \int_V \epsilon_{ijk}(\lambda v_{k,j})_{,i}\,dV \qquad \text{by (1.157)} \\ &= \int_V (\epsilon_{ijk}\lambda_{,i}v_{k,j} + \epsilon_{ijk}\lambda v_{k,ji})\,dV \\ &= \int_V \lambda_{,i}b_i\,dV \qquad \text{since } \lambda\epsilon_{ijk}v_{k,ji} = 0\end{aligned}$$

MISCELLANEOUS PROBLEMS

1.56. For the arbitrary vectors **a** and **b**, show that

$$\lambda = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\cdot\mathbf{b})^2 = (ab)^2$$

Interchange the dot and cross in the first term. Then

$$\begin{aligned}\lambda &= \mathbf{a}\cdot\mathbf{b}\times(\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b})\\ &= \mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{b})\mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{b}] + (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b})\\ &= (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{b}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b})\\ &= (ab)^2\end{aligned}$$

since the second and third terms cancel.

1.57. If $\dot{\mathbf{u}} = \boldsymbol{\omega}\times\mathbf{u}$ and $\dot{\mathbf{v}} = \boldsymbol{\omega}\times\mathbf{v}$, show that $\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \boldsymbol{\omega}\times(\mathbf{u}\times\mathbf{v})$.

(*a*) In symbolic notation,

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) &= \dot{\mathbf{u}}\times\mathbf{v} + \mathbf{u}\times\dot{\mathbf{v}} = (\boldsymbol{\omega}\times\mathbf{u})\times\mathbf{v} + \mathbf{u}\times(\boldsymbol{\omega}\times\mathbf{v})\\ &= (\mathbf{v}\cdot\boldsymbol{\omega})\mathbf{u} - (\mathbf{v}\cdot\mathbf{u})\boldsymbol{\omega} + (\mathbf{u}\cdot\mathbf{v})\boldsymbol{\omega} - (\mathbf{u}\cdot\boldsymbol{\omega})\mathbf{v}\\ &= (\mathbf{v}\cdot\boldsymbol{\omega})\mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\omega})\mathbf{v} = \boldsymbol{\omega}\times(\mathbf{u}\times\mathbf{v})\end{aligned}$$

(*b*) In indicial notation, let $\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \mathbf{w}$. Then

$$w_i = \frac{d}{dt}(\epsilon_{ijk}u_jv_k) = \epsilon_{ijk}\dot{u}_jv_k + \epsilon_{ijk}u_j\dot{v}_k$$

and since $\dot{u}_j = \epsilon_{jpq}\omega_pu_q$ and $\dot{v}_k = \epsilon_{kmn}\omega_mv_n$,

$$w_i = \epsilon_{ijk}\epsilon_{jpq}\omega_pu_qv_k + \epsilon_{ijk}\epsilon_{kmn}u_j\omega_mv_n = (\epsilon_{ijk}\epsilon_{kmn} - \epsilon_{ink}\epsilon_{kmj})u_j\omega_mv_n$$

and using the result of Problem *1.59*(*a*) below,

$$\begin{aligned}w_i &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} - \delta_{im}\delta_{jn} + \delta_{ij}\delta_{mn})u_j\omega_mv_n\\ &= (\delta_{ij}\delta_{mn} - \delta_{in}\delta_{jm})u_j\omega_mv_n = \epsilon_{imk}\epsilon_{kjn}u_j\omega_mv_n\end{aligned}$$

which is the indicial form of $\boldsymbol{\omega}\times(\mathbf{u}\times\mathbf{v})$.

1.58. Establish the identity $\epsilon_{pqs}\epsilon_{mnr} = \begin{vmatrix}\delta_{mp} & \delta_{mq} & \delta_{ms}\\ \delta_{np} & \delta_{nq} & \delta_{ns}\\ \delta_{rp} & \delta_{rq} & \delta_{rs}\end{vmatrix}$.

Let the determinant of A_{ij} be given by $\det A = \begin{vmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{vmatrix}$. An interchange of rows or columns causes a sign change. Thus

$$\begin{vmatrix}A_{21} & A_{22} & A_{23}\\ A_{11} & A_{12} & A_{13}\\ A_{31} & A_{32} & A_{33}\end{vmatrix} = \begin{vmatrix}A_{12} & A_{11} & A_{13}\\ A_{22} & A_{21} & A_{23}\\ A_{32} & A_{31} & A_{33}\end{vmatrix} = -\det A$$

and for an arbitrary number of row changes,

$$\begin{vmatrix}A_{m1} & A_{m2} & A_{m3}\\ A_{n1} & A_{n2} & A_{n3}\\ A_{r1} & A_{r2} & A_{r3}\end{vmatrix} = \epsilon_{mnr}\det A$$

or column changes,

$$\begin{vmatrix} A_{1p} & A_{1q} & A_{1s} \\ A_{2p} & A_{2q} & A_{2s} \\ A_{3p} & A_{3q} & A_{3s} \end{vmatrix} = \epsilon_{pqs} \det A$$

Hence for an arbitrary row and column interchange sequence,

$$\begin{vmatrix} A_{mp} & A_{mq} & A_{ms} \\ A_{np} & A_{nq} & A_{ns} \\ A_{rp} & A_{rq} & A_{rs} \end{vmatrix} = \epsilon_{mnr}\epsilon_{pqs} \det A$$

When $A_{ij} \equiv \delta_{ij}$, $\det A = 1$ and the identity is established.

1.59. Use the results of Problem 1.58 to prove (*a*) $\epsilon_{pqs}\epsilon_{snr} = \delta_{pn}\delta_{qr} - \delta_{pr}\delta_{qn}$, (*b*) $\epsilon_{pqs}\epsilon_{sqr} = -2\delta_{pr}$.

Expanding the determinant of the identity in Problem 1.58,

$$\epsilon_{pqs}\epsilon_{mnr} = \delta_{mp}(\delta_{nq}\delta_{rs} - \delta_{ns}\delta_{rq}) + \delta_{mq}(\delta_{ns}\delta_{rp} - \delta_{np}\delta_{rs}) + \delta_{ms}(\delta_{np}\delta_{rq} - \delta_{nq}\delta_{rp})$$

(*a*) Identifying s with m yields,

$$\begin{aligned} \epsilon_{pqs}\epsilon_{snr} &= \delta_{sp}(\delta_{nq}\delta_{rs} - \delta_{ns}\delta_{rq}) + \delta_{sq}(\delta_{ns}\delta_{rp} - \delta_{np}\delta_{rs}) + \delta_{ss}(\delta_{np}\delta_{rq} - \delta_{nq}\delta_{rp}) \\ &= \delta_{rp}\delta_{nq} - \delta_{pn}\delta_{rq} + \delta_{qn}\delta_{rp} - \delta_{np}\delta_{qr} + 3\delta_{np}\delta_{rq} - 3\delta_{nq}\delta_{rp} \\ &= \delta_{np}\delta_{rq} - \delta_{nq}\delta_{rp} \end{aligned}$$

(*b*) Identifying q with n in (*a*),

$$\epsilon_{pqs}\epsilon_{sqr} = \delta_{qp}\delta_{rq} - \delta_{qq}\delta_{rp} = \delta_{pr} - 3\delta_{pr} = -2\delta_{pr}$$

1.60. If the dyadic $\mathbf{B}$ is skew-symmetric $\mathbf{B} = -\mathbf{B}_c$, show that $\mathbf{B}_v \times \mathbf{a} = 2\mathbf{a}\cdot\mathbf{B}$.

Writing $\mathbf{B} = \mathbf{b}_1\hat{\mathbf{e}}_1 + \mathbf{b}_2\hat{\mathbf{e}}_2 + \mathbf{b}_3\hat{\mathbf{e}}_3$ (see Problem 1.6),

$$\mathbf{B}_v = \mathbf{b}_1 \times \hat{\mathbf{e}}_1 + \mathbf{b}_2 \times \hat{\mathbf{e}}_2 + \mathbf{b}_3 \times \hat{\mathbf{e}}_3$$

and

$$\begin{aligned} \mathbf{B}_v \times \mathbf{a} &= (\mathbf{b}_1 \times \hat{\mathbf{e}}_1) \times \mathbf{a} + (\mathbf{b}_2 \times \hat{\mathbf{e}}_2) \times \mathbf{a} + (\mathbf{b}_3 \times \hat{\mathbf{e}}_3) \times \mathbf{a} \\ &= (\mathbf{a}\cdot\mathbf{b}_1)\hat{\mathbf{e}}_1 - (\mathbf{a}\cdot\hat{\mathbf{e}}_1)\mathbf{b}_1 + (\mathbf{a}\cdot\mathbf{b}_2)\hat{\mathbf{e}}_2 - (\mathbf{a}\cdot\hat{\mathbf{e}}_2)\mathbf{b}_2 + (\mathbf{a}\cdot\mathbf{b}_3)\hat{\mathbf{e}}_3 - (\mathbf{a}\cdot\hat{\mathbf{e}}_3)\mathbf{b}_3 \\ &= \mathbf{a}\cdot(\mathbf{b}_1\hat{\mathbf{e}}_1 + \mathbf{b}_2\hat{\mathbf{e}}_2 + \mathbf{b}_3\hat{\mathbf{e}}_3) - \mathbf{a}\cdot(\hat{\mathbf{e}}_1\mathbf{b}_1 + \hat{\mathbf{e}}_2\mathbf{b}_2 + \hat{\mathbf{e}}_3\mathbf{b}_3) \\ &= \mathbf{a}\cdot\mathbf{B} - \mathbf{a}\cdot\mathbf{B}_c = 2\mathbf{a}\cdot\mathbf{B} \end{aligned}$$

1.61. Use the Hamilton-Cayley equation to obtain $(\mathbf{B})^4$ for the tensor $\mathbf{B} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & -2 \end{pmatrix}$. Check the result directly by squaring $(\mathbf{B})^2$.

The characteristic equation for $\mathbf{B}$ is given by

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 3-\lambda & 0 \\ -1 & 0 & -2-\lambda \end{vmatrix} = -(\lambda^3 - 2\lambda^2 - 6\lambda + 9) = 0$$

By the Hamilton-Cayley theorem the tensor satisfies its own characteristic equation. Hence $(\mathbf{B})^3 - 2(\mathbf{B})^2 - 6\mathbf{B} + 9\mathbf{I} = 0$, and multiplying this equation by $\mathbf{B}$ yields $(\mathbf{B})^4 = 2(\mathbf{B})^3 + 6(\mathbf{B})^2 - 9\mathbf{B}$ or $(\mathbf{B})^4 = 10(\mathbf{B})^2 + 3\mathbf{B} - 18\mathbf{I}$. Hence

$$(\mathbf{B})^4 \;=\; 10\begin{pmatrix}2 & 0 & 1\\ 0 & 9 & 0\\ 1 & 0 & 5\end{pmatrix} + \begin{pmatrix}3 & 0 & -3\\ 0 & 9 & 0\\ -3 & 0 & -6\end{pmatrix} - \begin{pmatrix}18 & 0 & 0\\ 0 & 18 & 0\\ 0 & 0 & 18\end{pmatrix} \;=\; \begin{pmatrix}5 & 0 & 7\\ 0 & 81 & 0\\ 7 & 0 & 26\end{pmatrix}$$

Checking by direct matrix multiplication of $(\mathbf{B})^2$,

$$(\mathcal{B})^4 \;=\; \begin{bmatrix}2 & 0 & 1\\ 0 & 9 & 0\\ 1 & 0 & 5\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 0 & 9 & 0\\ 1 & 0 & 5\end{bmatrix} \;=\; \begin{bmatrix}5 & 0 & 7\\ 0 & 81 & 0\\ 7 & 0 & 26\end{bmatrix}$$

1.62. Prove that (*a*) A_{ii}, (*b*) $A_{ij}A_{ij}$, (*c*) $\epsilon_{ijk}\epsilon_{kjp}A_{ip}$ are invariant under the coordinate transformation represented by (*1.103*), i.e. show that $A_{ii} = A'_{ii}$, etc.

(*a*) By (*1.103*), $A_{ij} = a_{pi}a_{qj}A'_{pq}$. Hence $A_{ii} = a_{pi}a_{qi}A'_{pq} = \delta_{pq}A'_{pq} = A'_{pp} = A'_{ii}$.

(*b*) $A_{ij}A_{ij} \;=\; a_{pi}a_{qj}A'_{pq}a_{mi}a_{nj}A'_{mn} \;=\; \delta_{pm}\delta_{qn}A'_{pq}A'_{mn} \;=\; A'_{pq}A'_{pq} \;=\; A'_{ij}A'_{ij}$

(*c*)
$$\begin{aligned}\epsilon_{ijk}\epsilon_{kjp}A_{ip} &= \epsilon_{ijk}\epsilon_{kjp}a_{mi}a_{np}A'_{mn} \;=\; (\delta_{ij}\delta_{jp} - \delta_{ip}\delta_{jj})a_{mi}a_{np}A'_{mn}\\ &= (\delta_{mn} - \delta_{mn}\delta_{jj})A'_{mn} \;=\; (\delta_{mj}\delta_{nj} - \delta_{mn}\delta_{jj})A'_{mn} \;=\; \epsilon_{mjk}\epsilon_{kjn}A'_{mn}\end{aligned}$$

1.63. Show that the dual vector of the arbitrary tensor T_{ij} depends only upon $T_{[ij]}$ but that the product $T_{ij}S_{ij}$ of T_{ij} with the symmetric tensor S_{ij} is independent of $T_{[ij]}$.

By (*1.110*) the dual vector of T_{ij} is $v_i = \epsilon_{ijk}T_{jk}$, or $v_i = \epsilon_{ijk}(T_{(jk)} + T_{[jk]}) = \epsilon_{ijk}T_{[jk]}$ since $\epsilon_{ijk}T_{(jk)} = 0$ (ϵ_{ijk} is anti-symmetric in j and k, $T_{(jk)}$ symmetric in j and k).

For the product $T_{ij}S_{ij} = T_{(ij)}S_{ij} + T_{[ij]}S_{ij}$. Here $T_{[ij]}S_{ij} = 0$ and $T_{ij}S_{ij} = T_{(ij)}S_{ij}$.

1.64. Show that $\mathbf{D}:\mathbf{E}$ is equal to $\mathbf{D}\cdot\cdot\mathbf{E}$ if $\mathbf{E}$ is a symmetric dyadic.

Write $\mathbf{D} = D_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$ and $\mathbf{E} = E_{pq}\hat{\mathbf{e}}_p\hat{\mathbf{e}}_q$. By (*1.31*), $\mathbf{D}:\mathbf{E} = D_{ij}E_{pq}(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_p)(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_q)$. By (*1.35*), $\mathbf{D}\cdot\cdot\mathbf{E} = D_{ij}E_{pq}(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_p)(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_q) = D_{ij}E_{qp}(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_p)(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_q)$ since $E_{pq} = E_{qp}$. Now interchanging the role of dummy indices p and q in this last expression, $\mathbf{D}\cdot\cdot\mathbf{E} = D_{ij}E_{pq}(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_q)(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_p)$.

1.65. Use the indicial notation to prove the vector identity $\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\mathbf{a} - \mathbf{b}(\nabla\cdot\mathbf{a}) + \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{a}\cdot\nabla\mathbf{b}$.

Let $\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{v}$; then $v_p = \epsilon_{pqi}\epsilon_{ijk}\partial_q a_j b_k$ or

$$\begin{aligned}v_p &= \epsilon_{pqi}\epsilon_{ijk}(a_jb_k)_{,q} \;=\; \epsilon_{pqi}\epsilon_{ijk}(a_{j,q}b_k + a_jb_{k,q})\\ &= (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})(a_{j,q}b_k + a_jb_{k,q}) \;=\; a_{p,q}b_q - a_{q,q}b_p + a_pb_{q,q} - a_qb_{p,q}\end{aligned}$$

Hence $\mathbf{v} = \mathbf{b}\cdot\nabla\mathbf{a} - \mathbf{b}(\nabla\cdot\mathbf{a}) + \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{a}\cdot\nabla\mathbf{b}$.

1.66. By means of the divergence theorem of Gauss show that $\int_S \hat{\mathbf{n}}\times(\mathbf{a}\times\mathbf{x})\,dS = 2\mathbf{a}V$ where V is the volume enclosed by the surface S having the outward normal $\mathbf{n}$. The position vector to any point in V is $\mathbf{x}$, and $\mathbf{a}$ is an arbitrary constant vector.

In the indicial notation the surface integral is $\int_S \epsilon_{qpi}n_p\epsilon_{ijk}a_jx_k\,dS$. By (*1.157*) this becomes the volume integral $\int_V (\epsilon_{qpi}\epsilon_{ijk}a_jx_k)_{,p}\,dV$ and since $\mathbf{a}$ is constant, the last expression becomes

$$\begin{aligned}\int_V \epsilon_{qpi}\epsilon_{ijk}a_jx_{k,p}\,dV &= \int_V (\delta_{qj}\delta_{pk} - \delta_{qk}\delta_{pj})a_jx_{k,p}\,dV \;=\; \int_V (a_qx_{p,p} - a_px_{q,p})\,dV\\ &= \int_V (a_q\delta_{pp} - a_p\delta_{qp})\,dV \;=\; \int_V (3a_q - a_q)\,dV \;=\; 2a_qV\end{aligned}$$

1.67. For the reflection of axes shown in Fig. 1-21 show that the transformation is orthogonal.

From the figure the transformation matrix is

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The orthogonality conditions $a_{ij}a_{ik} = \delta_{jk}$ or $a_{ji}a_{ki} = \delta_{jk}$ are clearly satisfied. In matrix form, by *(1.117)*,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

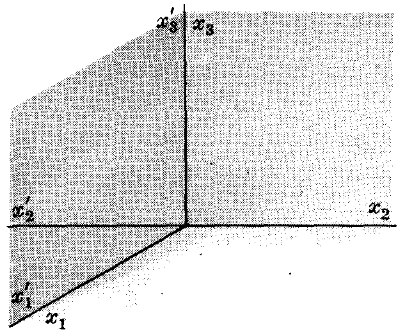


Fig. 1-21

1.68. Show that $(\mathbf{I} \times \mathbf{v}) \cdot \mathbf{D} = \mathbf{v} \times \mathbf{D}$.

$$\begin{aligned} \mathbf{I} \times \mathbf{v} &= (\hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}}) \times (v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) \\ &= \hat{\mathbf{i}}(v_y\hat{\mathbf{k}} - v_z\hat{\mathbf{j}}) + \hat{\mathbf{j}}(-v_x\hat{\mathbf{k}} + v_z\hat{\mathbf{i}}) + \hat{\mathbf{k}}(v_x\hat{\mathbf{j}} - v_y\hat{\mathbf{i}}) \\ &= (\mathbf{v} \times \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{v} \times \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{v} \times \hat{\mathbf{k}})\hat{\mathbf{k}} = \mathbf{v} \times \mathbf{I} \end{aligned}$$

Hence $(\mathbf{I} \times \mathbf{v}) \cdot \mathbf{D} = \mathbf{v} \times \mathbf{I} \cdot \mathbf{D} = \mathbf{v} \times \mathbf{D}$.

Supplementary Problems

1.69. Show that $\mathbf{u} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ are perpendicular to one another. Determine $\hat{\mathbf{w}}$ so that $\mathbf{u}, \mathbf{v}, \hat{\mathbf{w}}$ forms a right-handed triad. *Ans.* $\mathbf{w} = (-1/\sqrt{6})(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

1.70. Determine the transformation matrix between the $\mathbf{u}, \mathbf{v}, \hat{\mathbf{w}}$ axes of Problem 1.69 and the coordinate directions.

Ans. $$[a_{ij}] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$$

1.71. Use indicial notation to prove (*a*) $\nabla \cdot \mathbf{x} = 3$, (*b*) $\nabla \times \mathbf{x} = 0$, (*c*) $\mathbf{a} \cdot \nabla\mathbf{x} = \mathbf{a}$ where $\mathbf{x}$ is the position vector and $\mathbf{a}$ is a constant vector.

1.72. Determine the principal values of the symmetric part of the tensor $T_{ij} = \begin{pmatrix} 5 & -1 & 3 \\ 1 & -6 & -6 \\ -3 & -18 & 1 \end{pmatrix}$.
Ans. $\lambda_{(1)} = -15,\ \lambda_{(2)} = 5,\ \lambda_{(3)} = 10$

1.73. For the symmetric tensor $T_{ij} = \begin{pmatrix} 7 & 3 & 0 \\ 3 & 7 & 4 \\ 0 & 4 & 7 \end{pmatrix}$ determine the principal values and the directions of the principal axes.

Ans. $\lambda_{(1)} = 2,\ \lambda_{(2)} = 7,\ \lambda_{(3)} = 12,$

	x_1	x_2	x_3
x_1^*	$-3/5\sqrt{2}$	$1/\sqrt{2}$	$-4/5\sqrt{2}$
x_2^*	$4/5$	0	$-3/5$
x_3^*	$3/5\sqrt{2}$	$1/\sqrt{2}$	$4/5\sqrt{2}$

1.74. Given the arbitrary vector v and any unit vector $\hat{\mathbf{e}}$, show that v may be resolved into a component parallel and a component perpendicular to $\hat{\mathbf{e}}$, i.e. $\mathbf{v} = (\mathbf{v}\cdot\hat{\mathbf{e}})\hat{\mathbf{e}} + \hat{\mathbf{e}} \times (\mathbf{v} \times \hat{\mathbf{e}})$.

1.75. If $\nabla \cdot \mathbf{v} = 0$, $\nabla \times \mathbf{v} = \dot{\mathbf{w}}$ and $\nabla \times \mathbf{w} = -\dot{\mathbf{v}}$, show that $\nabla^2 \mathbf{v} = \ddot{\mathbf{v}}$.

1.76. Check the result of Problem 1.48 by direct multiplication to show that $\sqrt{\mathsf{T}}\sqrt{\mathsf{T}} = \mathsf{T}$.

1.77. Determine the square root of the tensor $\mathsf{B} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$.

Ans. $\sqrt{\mathsf{B}} = \begin{pmatrix} \frac{1}{2}(\sqrt{5}+1) & \frac{1}{2}(\sqrt{5}-1) & 0 \\ \frac{1}{2}(\sqrt{5}-1) & \frac{1}{2}(\sqrt{5}+1) & 0 \\ 0 & 0 & 3 \end{pmatrix}$

1.78. Using the result of Problem 1.40, $\det \mathsf{A} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$, show that $\det(\mathsf{AB}) = \det \mathsf{A} \det \mathsf{B}$.

1.79. Verify that (*a*) $\delta_{3p} v_p = v_3$, (*b*) $\delta_{3i} A_{ji} = A_{j3}$, (*c*) $\delta_{ij}\epsilon_{ijk} = 0$, (*d*) $\delta_{i2}\delta_{j3} A_{ij} = A_{23}$.

1.80. Let the axes $Ox_1'x_2'x_3'$ be related to $Ox_1x_2x_3$ by the table

	x_1	x_2	x_3
x_1'	$3/5\sqrt{2}$	$1/\sqrt{2}$	$4/5\sqrt{2}$
x_2'	$4/5$	0	$-3/5$
x_3'	$-3/5\sqrt{2}$	$1/\sqrt{2}$	$-4/5\sqrt{2}$

(*a*) Show that the orthogonality conditions $a_{ij}a_{ik} = \delta_{jk}$ and $a_{pq}a_{sq} = \delta_{ps}$ are satisfied.

(*b*) What are the primed coordinates of the point having position vector $\mathbf{x} = 2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3$?

(*c*) What is the equation of the plane $x_1 - x_2 + 3x_3 = 1$ in the primed system?

Ans. (*b*) $(2/5\sqrt{2},\ 11/5,\ -2/5\sqrt{2})$ (*c*) $\sqrt{2}\,x_1' - x_2' - 2\sqrt{2}\,x_3' = 1$

1.81. Show that the volume V enclosed by the surface S may be given as $V = \frac{1}{6}\int_S \nabla(\mathbf{x}\cdot\mathbf{x})\cdot\hat{\mathbf{n}}\,dS$ where x is the position vector and n the unit normal to the surface. *Hint*: Write $V = (1/6)\int_S (x_i x_i)_{,j} n_j \, dS$ and use (*1.157*).

Chapter 2

Analysis of Stress

2.1 THE CONTINUUM CONCEPT

The molecular nature of the structure of matter is well established. In numerous investigations of material behavior, however, the individual molecule is of no concern and only the behavior of the material as a whole is deemed important. For these cases the observed macroscopic behavior is usually explained by disregarding molecular considerations and, instead, by assuming the material to be continuously distributed throughout its volume and to completely fill the space it occupies. This *continuum concept* of matter is the fundamental postulate of Continuum Mechanics. Within the limitations for which the continuum assumption is valid, this concept provides a framework for studying the behavior of solids, liquids and gases alike.

Adoption of the continuum viewpoint as the basis for the mathematical description of material behavior means that field quantities such as stress and displacement are expressed as piecewise continuous functions of the space coordinates and time.

2.2 HOMOGENEITY. ISOTROPY. MASS-DENSITY

A *homogeneous* material is one having identical properties at all points. With respect to some property, a material is *isotropic* if that property is the same in all directions at a point. A material is called *anisotropic* with respect to those properties which are directional at a point.

The concept of *density* is developed from the *mass-volume ratio* in the neighborhood of a point in the continuum. In Fig. 2-1 the mass in the small element of volume ΔV is denoted by ΔM. The *average density* of the material within ΔV is therefore

$$\rho_{(av)} = \frac{\Delta M}{\Delta V} \qquad (2.1)$$

The *density* at some interior point P of the volume element ΔV is given mathematically in accordance with the continuum concept by the limit,

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV} \qquad (2.2)$$

Mass-density ρ is a scalar quantity.

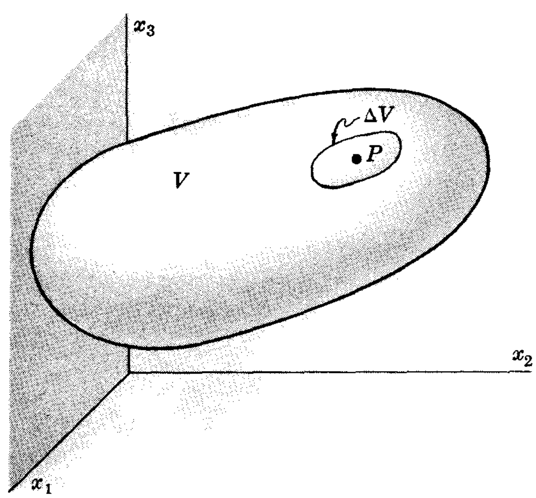


Fig. 2-1

2.3 BODY FORCES. SURFACE FORCES

Forces are vector quantities which are best described by intuitive concepts such as push or pull. Those forces which act on all elements of volume of a continuum are known as *body forces*. Examples are gravity and inertia forces. These forces are represented by the symbol b_i (force per unit mass), or as p_i (force per unit volume). They are related through the density by the equation

$$\rho b_i = p_i \quad \text{or} \quad \rho \mathbf{b} = \mathbf{p} \tag{2.3}$$

Those forces which act on a surface element, whether it is a portion of the bounding surface of the continuum or perhaps an arbitrary internal surface, are known as *surface forces*. These are designated by f_i (force per unit area). Contact forces between bodies are a type of surface forces.

2.4 CAUCHY'S STRESS PRINCIPLE. THE STRESS VECTOR

A material continuum occupying the region R of space, and subjected to surface forces f_i and body forces b_i, is shown in Fig. 2-2. As a result of forces being transmitted from one portion of the continuum to another, the material within an arbitrary volume V enclosed by the surface S interacts with the material outside of this volume. Taking n_i as the outward unit normal at point P of a small element of surface ΔS of S, let Δf_i be the resultant force exerted across ΔS upon the material within V by the material outside of V. Clearly the force element Δf_i will depend upon the choice of ΔS and upon n_i. It should also be noted that the distribution of force on ΔS is not necessarily uniform. Indeed the force distribution is, in general, equipollent to a force and a moment at P, as shown in Fig. 2-2 by the vectors Δf_i and ΔM_i.

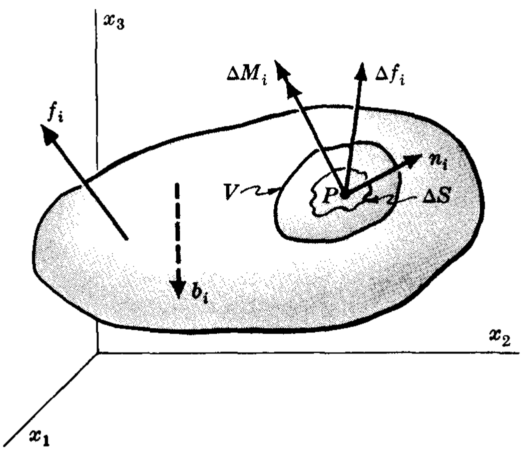


Fig. 2-2

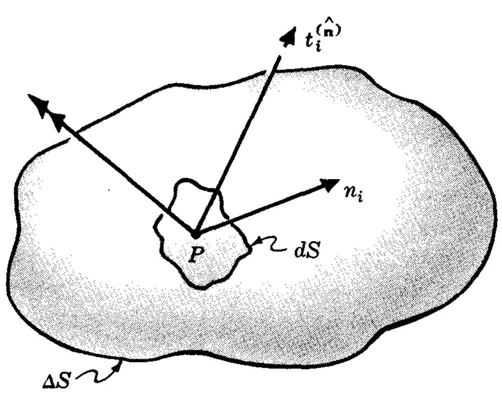


Fig. 2-3

The average force per unit area on ΔS is given by $\Delta f_i/\Delta S$. The *Cauchy stress principle* asserts that this ratio $\Delta f_i/\Delta s$ tends to a definite limit df_i/dS as ΔS approaches zero at the point P, while at the same time the moment of Δf_i about the point P vanishes in the limiting process. The resulting vector df_i/dS (force per unit area) is called the *stress vector* $t_i^{(\hat{n})}$ and is shown in Fig. 2-3. If the moment at P were not to vanish in the limiting process, a *couple-stress vector*, shown by the double-headed arrow in Fig. 2-3, would also be defined at the point. One branch of the theory of elasticity considers such couple stresses but they are not considered in this text.

Mathematically the stress vector is defined by

$$t_i^{(\hat{\mathbf{n}})} = \lim_{\Delta S \to 0} \frac{\Delta f_i}{\Delta S} = \frac{df_i}{dS} \quad \text{or} \quad \mathbf{t}^{(\hat{\mathbf{n}})} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{f}}{\Delta S} = \frac{d\mathbf{f}}{dS} \tag{2.4}$$

The notation $t_i^{(\hat{\mathbf{n}})}$ (or $\mathbf{t}^{(\hat{\mathbf{n}})}$) is used to emphasize the fact that the stress vector at a given point P in the continuum depends explicitly upon the particular surface element ΔS chosen there, as represented by the unit normal n_i (or $\hat{\mathbf{n}}$). For some differently oriented surface element, having a different unit normal, the associated stress vector at P will also be different. The stress vector arising from the action across ΔS at P of the material within V upon the material outside is the vector $-t_i^{(\hat{\mathbf{n}})}$. Thus by Newton's law of action and reaction,

$$-t_i^{(\hat{\mathbf{n}})} = t_i^{(-\hat{\mathbf{n}})} \quad \text{or} \quad -\mathbf{t}^{(\hat{\mathbf{n}})} = \mathbf{t}^{(-\hat{\mathbf{n}})} \tag{2.5}$$

The *stress vector* is very often referred to as the *traction vector.*

2.5 STATE OF STRESS AT A POINT. STRESS TENSOR

At an arbitrary point P in a continuum, Cauchy's stress principle associates a stress vector $t_i^{(\hat{\mathbf{n}})}$ with each unit normal vector n_i, representing the orientation of an infinitesimal surface element having P as an interior point. This is illustrated in Fig. 2-3. The totality of all possible pairs of such vectors $t_i^{(\hat{\mathbf{n}})}$ and n_i at P defines the *state of stress* at that point. Fortunately it is not necessary to specify every pair of stress and normal vectors to completely describe the state of stress at a given point. This may be accomplished by giving the stress vector on each of three mutually perpendicular planes at P. Coordinate transformation equations then serve to relate the stress vector on any other plane at the point to the given three.

Adopting planes perpendicular to the coordinate axes for the purpose of specifying the state of stress at a point, the appropriate stress and normal vectors are shown in Fig. 2-4.

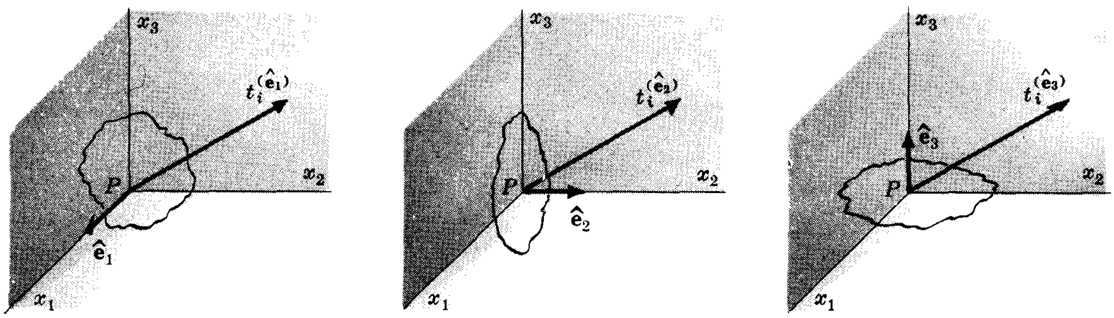


Fig. 2-4

For convenience, the three separate diagrams in Fig. 2-4 are often combined into a single schematic representation as shown in Fig. 2-5 below.

Each of the three coordinate-plane stress vectors may be written according to (*1.69*) in terms of its Cartesian components as

$$\begin{aligned} \mathbf{t}^{(\hat{\mathbf{e}}_1)} &= t_1^{(\hat{\mathbf{e}}_1)}\hat{\mathbf{e}}_1 + t_2^{(\hat{\mathbf{e}}_1)}\hat{\mathbf{e}}_2 + t_3^{(\hat{\mathbf{e}}_1)}\hat{\mathbf{e}}_3 = t_j^{(\hat{\mathbf{e}}_1)}\hat{\mathbf{e}}_j \\ \mathbf{t}^{(\hat{\mathbf{e}}_2)} &= t_1^{(\hat{\mathbf{e}}_2)}\hat{\mathbf{e}}_1 + t_2^{(\hat{\mathbf{e}}_2)}\hat{\mathbf{e}}_2 + t_3^{(\hat{\mathbf{e}}_2)}\hat{\mathbf{e}}_3 = t_j^{(\hat{\mathbf{e}}_2)}\hat{\mathbf{e}}_j \\ \mathbf{t}^{(\hat{\mathbf{e}}_3)} &= t_1^{(\hat{\mathbf{e}}_3)}\hat{\mathbf{e}}_1 + t_2^{(\hat{\mathbf{e}}_3)}\hat{\mathbf{e}}_2 + t_3^{(\hat{\mathbf{e}}_3)}\hat{\mathbf{e}}_3 = t_j^{(\hat{\mathbf{e}}_3)}\hat{\mathbf{e}}_j \end{aligned} \tag{2.6}$$

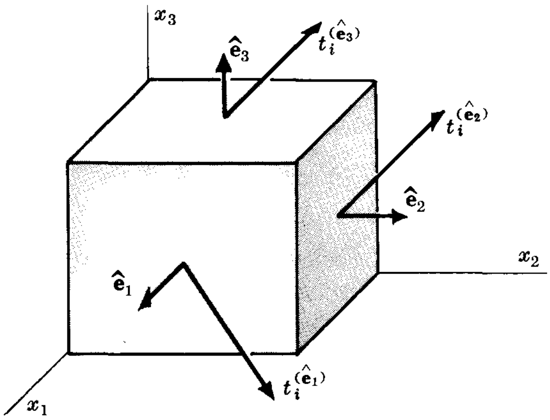


Fig. 2-5

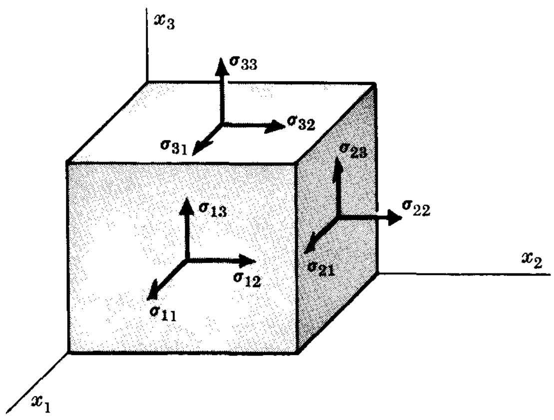


Fig. 2-6

The nine stress vector components,

$$t_j^{(\hat{e}_i)} \equiv \sigma_{ij} \tag{2.7}$$

are the components of a second-order Cartesian tensor known as the *stress tensor.* The equivalent stress dyadic is designated by $\mathbf{\Sigma}$, so that explicit component and matrix representations of the stress tensor, respectively, take the forms

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \text{or} \quad [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.8}$$

Pictorially, the stress tensor components may be displayed with reference to the coordinate planes as shown in Fig. 2-6. The components perpendicular to the planes $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ are called *normal stresses.* Those acting in (tangent to) the planes $(\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \sigma_{32})$ are called *shear stresses.* A stress component is *positive* when it acts in the positive direction of the coordinate axes, and on a plane whose outer normal points in one of the positive coordinate directions. The component σ_{ij} acts in the direction of the jth coordinate axis and on the plane whose outward normal is parallel to the ith coordinate axis. The stress components shown in Fig. 2-6 are all positive.

2.6 THE STRESS TENSOR — STRESS VECTOR RELATIONSHIP

The relationship between the stress tensor σ_{ij} at a point P and the stress vector $t_i^{(\hat{n})}$ on a plane of arbitrary orientation at that point may be established through the force equilibrium or momentum balance of a small tetrahedron of the continuum, having its vertex at P. The base of the tetrahedron is taken perpendicular to n_i, and the three faces are taken perpendicular to the coordinate planes as shown by Fig. 2-7. Designating the area of the base ABC as dS, the areas of the faces are the projected areas, $dS_1 = dS\,n_1$ for face CPB, $dS_2 = dS\,n_2$ for face APC, $dS_3 = dS\,n_3$ for face BPA or

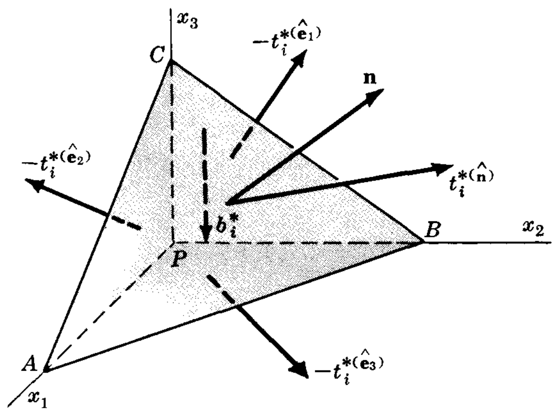


Fig. 2-7

$$dS_i = dS(\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_i) = dS\cos(\hat{\mathbf{n}},\hat{\mathbf{e}}_i) = dS\,n_i \qquad (2.9)$$

The average traction vectors $-t_i^{*(\hat{\mathbf{e}}_j)}$ on the faces and $t_i^{*(\hat{\mathbf{n}})}$ on the base, together with the average body forces (including inertia forces, if present), acting on the tetrahedron are shown in the figure. Equilibrium of forces on the tetrahedron requires that

$$t_i^{*(\hat{\mathbf{n}})}\,dS - t_i^{*(\hat{\mathbf{e}}_1)}\,dS_1 - t_i^{*(\hat{\mathbf{e}}_2)}\,dS_2 - t_i^{*(\hat{\mathbf{e}}_3)}\,dS_3 + \rho b_i^*\,dV = 0 \qquad (2.10)$$

If now the linear dimensions of the tetrahedron are reduced in a constant ratio to one another, the body forces, being an order higher in the small dimensions, tend to zero more rapidly than the surface forces. At the same time, the average stress vectors approach the specific values appropriate to the designated directions at P. Therefore by this limiting process and the substitution (*2.9*), equation (*2.10*) reduces to

$$t_i^{(\hat{\mathbf{n}})}\,dS = t_i^{(\hat{\mathbf{e}}_1)}n_1\,dS + t_i^{(\hat{\mathbf{e}}_2)}n_2\,dS + t_i^{(\hat{\mathbf{e}}_3)}n_3\,dS = t_i^{(\hat{\mathbf{e}}_j)}n_j\,dS \qquad (2.11)$$

Cancelling the common factor dS and using the identity $t_i^{(\hat{\mathbf{e}}_j)} \equiv \sigma_{ji}$, (*2.11*) becomes

$$t_i^{(\hat{\mathbf{n}})} = \sigma_{ji}n_j \quad \text{or} \quad \mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}}\cdot\mathbf{\Sigma} \qquad (2.12)$$

Equation (*2.12*) is also often expressed in the matrix form

$$[t_{1j}^{(\hat{\mathbf{n}})}] = [n_{1k}][\sigma_{kj}] \qquad (2.13)$$

which is written explicitly

$$[t_1^{(\hat{\mathbf{n}})}, t_2^{(\hat{\mathbf{n}})}, t_3^{(\hat{\mathbf{n}})}] = [n_1, n_2, n_3]\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} \qquad (2.14)$$

The matrix form (*2.14*) is equivalent to the component equations

$$\begin{aligned} t_1^{(\hat{\mathbf{n}})} &= n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31}\\ t_2^{(\hat{\mathbf{n}})} &= n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32}\\ t_3^{(\hat{\mathbf{n}})} &= n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33} \end{aligned} \qquad (2.15)$$

2.7 FORCE AND MOMENT EQUILIBRIUM. STRESS TENSOR SYMMETRY

Equilibrium of an arbitrary volume V of a continuum, subjected to a system of surface forces $t_i^{(\hat{\mathbf{n}})}$ and body forces b_i (including inertia forces, if present) as shown in Fig. 2-8, requires that the resultant force and moment acting on the volume be zero.

Summation of surface and body forces results in the integral relation,

$$\int_S t_i^{(\hat{\mathbf{n}})}\,dS + \int_V \rho b_i\,dV = 0$$

or $\qquad (2.16)$

$$\int_S \mathbf{t}^{(\hat{\mathbf{n}})}\,dS + \int_V \rho\mathbf{b}\,dV = 0$$

Replacing $t_i^{(\hat{\mathbf{n}})}$ here by $\sigma_{ji}n_j$ and converting the resulting surface integral to a volume integral by the divergence theorem of Gauss (*1.157*), equation (*2.16*) becomes

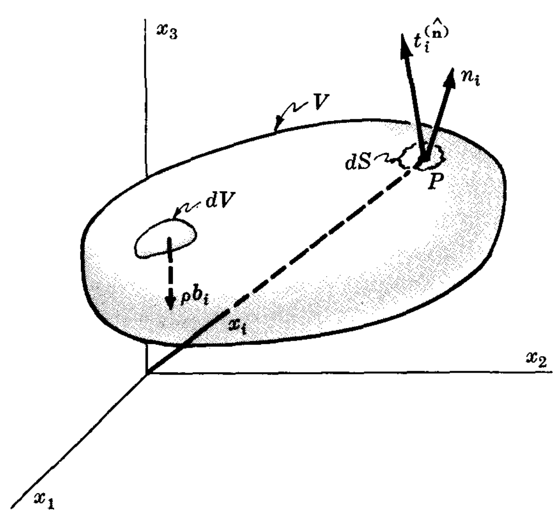


Fig. 2-8

$$\int_V (\sigma_{ji,j} + \rho b_i)\,dV = 0 \quad \text{or} \quad \int_V (\nabla \cdot \mathbf{\Sigma} + \rho \mathbf{b})\,dV = 0 \tag{2.17}$$

Since the volume V is arbitrary, the integrand in (*2.17*) must vanish, so that

$$\sigma_{ji,j} + \rho b_i = 0 \quad \text{or} \quad \nabla \cdot \mathbf{\Sigma} + \rho \mathbf{b} = 0 \tag{2.18}$$

which are called the *equilibrium equations.*

In the absence of distributed moments or couple-stresses, the equilibrium of moments about the origin requires that

$$\int_S \epsilon_{ijk} x_j t_k^{(\hat{\mathbf{n}})}\,dS + \int_V \epsilon_{ijk} x_j \rho b_k\,dV = 0$$

or

$$\int_S \mathbf{x} \times \mathbf{t}^{(\hat{\mathbf{n}})}\,dS + \int_V \mathbf{x} \times \rho \mathbf{b}\,dV = 0 \tag{2.19}$$

in which x_i is the position vector of the elements of surface and volume. Again, making the substitution $t_i^{(\hat{\mathbf{n}})} = \sigma_{ji} n_j$, applying the theorem of Gauss and using the result expressed in (*2.18*), the integrals of (*2.19*) are combined and reduced to

$$\int_V \epsilon_{ijk}\sigma_{jk}\,dV = 0 \quad \text{or} \quad \int_V \mathbf{\Sigma}_v\,dV = 0 \tag{2.20}$$

For the arbitrary volume V, (*2.20*) requires

$$\epsilon_{ijk}\sigma_{jk} = 0 \quad \text{or} \quad \mathbf{\Sigma}_v = 0 \tag{2.21}$$

Equation (*2.21*) represents the equations $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$, or in all

$$\sigma_{ij} = \sigma_{ji} \tag{2.22}$$

which shows that the *stress tensor is symmetric.* In view of (*2.22*), the equilibrium equations (*2.18*) are often written

$$\sigma_{ij,j} + \rho b_i = 0 \tag{2.23}$$

which appear in expanded form as

$$\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 &= 0 \\
\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 &= 0 \\
\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 &= 0
\end{aligned} \tag{2.24}$$

2.8 STRESS TRANSFORMATION LAWS

At the point P let the rectangular Cartesian coordinate systems $Px_1x_2x_3$ and $Px_1'x_2'x_3'$ of Fig. 2-9 be related to one another by the table of direction cosines

	x_1	x_2	x_3
x_1'	a_{11}	a_{12}	a_{13}
x_2'	a_{21}	a_{22}	a_{23}
x_3'	a_{31}	a_{32}	a_{33}

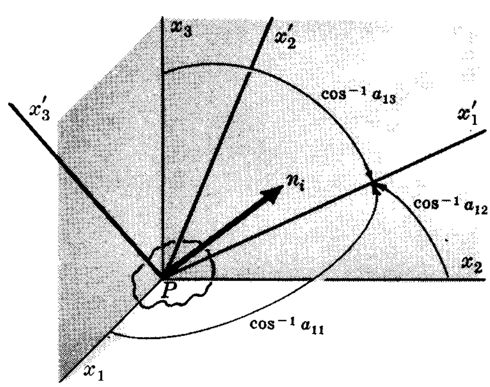


Fig. 2-9

or by the equivalent alternatives, the transformation matrix $[a_{ij}]$, or the transformation dyadic

$$\mathbf{A} = a_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \tag{2.25}$$

According to the transformation law for Cartesian tensors of order one (*1.93*), the components of the stress vector $t_i^{(\hat{n})}$ referred to the unprimed axes are related to the primed axes components $t_i'^{(\hat{n})}$ by the equation

$$t_i'^{(\hat{n})} = a_{ij}t_j^{(\hat{n})} \quad \text{or} \quad \mathbf{t}'^{(\hat{n})} = \mathbf{A}\cdot\mathbf{t}^{(\hat{n})} \tag{2.26}$$

Likewise, by the transformation law (*1.102*) for second-order Cartesian tensors, the stress tensor components in the two systems are related by

$$\sigma'_{ij} = a_{ip}a_{jq}\sigma_{pq} \quad \text{or} \quad \mathbf{\Sigma}' = \mathbf{A}\cdot\mathbf{\Sigma}\cdot\mathbf{A}_c \tag{2.27}$$

In matrix form, the stress vector transformation is written

$$[t_{i1}'^{(\hat{n})}] = [a_{ij}][t_{j1}^{(\hat{n})}] \tag{2.28}$$

and the stress tensor transformation as

$$[\sigma_{ij}] = [a_{ip}][\sigma_{pq}][a_{qj}] \tag{2.29}$$

Explicitly, the matrix multiplications in (*2.28*) and (*2.29*) are given respectively by

$$\begin{bmatrix} t_1'^{(\hat{n})} \\ t_2'^{(\hat{n})} \\ t_3'^{(\hat{n})} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} t_1^{(\hat{n})} \\ t_2^{(\hat{n})} \\ t_3^{(\hat{n})} \end{bmatrix} \tag{2.30}$$

and

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{2.31}$$

2.9 STRESS QUADRIC OF CAUCHY

At the point P in a continuum, let the stress tensor have the values σ_{ij} when referred to directions parallel to the local Cartesian axes $P\zeta_1\zeta_2\zeta_3$ shown in Fig. 2-10. The equation

$$\sigma_{ij}\zeta_i\zeta_j = \pm k^2 \quad \text{(a constant)} \tag{2.32}$$

represents geometrically similar quadric surfaces having a common center at P. The plus or minus choice assures the surfaces are real.

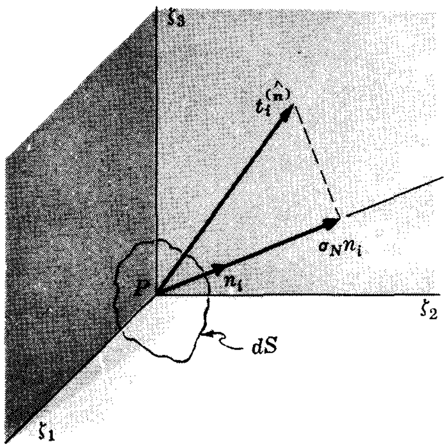


Fig. 2-10

The position vector $\mathbf{r}$ of an arbitrary point lying on the quadric surface has components $\zeta_i = rn_i$, where n_i is the unit normal in the direction of $\mathbf{r}$. At the point P the normal component $\sigma_N n_i$ of the stress vector $t_i^{(\hat{n})}$ has a magnitude

$$\sigma_N = t_i^{(\hat{n})}n_i = \mathbf{t}^{(\hat{n})}\cdot\mathbf{n} = \sigma_{ij}n_in_j \tag{2.33}$$

Accordingly if the constant k^2 of (*2.32*) is set equal to $\sigma_N r^2$, the resulting quadric

$$\sigma_{ij}\zeta_i\zeta_j = \pm\sigma_N r^2 \tag{2.34}$$

is called the *stress quadric of Cauchy*. From this definition it follows that the magnitude σ_N of the normal stress component on the surface element dS perpendicular to the position vector $\mathbf{r}$ of a point on Cauchy's stress quadric, is inversely proportional to r^2, i.e. $\sigma_N = \pm k^2/r^2$. Furthermore it may be shown that the stress vector $t_i^{(\hat{n})}$ acting on dS at P is parallel to the normal of the tangent plane of the Cauchy quadric at the point identified by $\mathbf{r}$.

2.10 PRINCIPAL STRESSES. STRESS INVARIANTS. STRESS ELLIPSOID

At the point P for which the stress tensor components are σ_{ij}, the equation (*2.12*), $t_i^{(\hat{n})} = \sigma_{ji} n_j$, associates with each direction n_i a stress vector $t_i^{(\hat{n})}$. Those directions for which $t_i^{(\hat{n})}$ and n_i are collinear as shown in Fig. 2-11 are called *principal stress directions*. For a principal stress direction,

$$t_i^{(\hat{n})} = \sigma n_i \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \sigma \hat{\mathbf{n}} \tag{2.35}$$

in which σ, the magnitude of the stress vector, is called a *principal stress value*. Substituting (*2.35*) into (*2.12*) and making use of the identities $n_i = \delta_{ij} n_j$ and $\sigma_{ij} = \sigma_{ji}$, results in the equations

$$(\sigma_{ij} - \delta_{ij}\sigma) n_j = 0 \quad \text{or} \quad (\mathbf{\Sigma} - \mathbf{I}\sigma)\cdot\hat{\mathbf{n}} = 0 \tag{2.36}$$

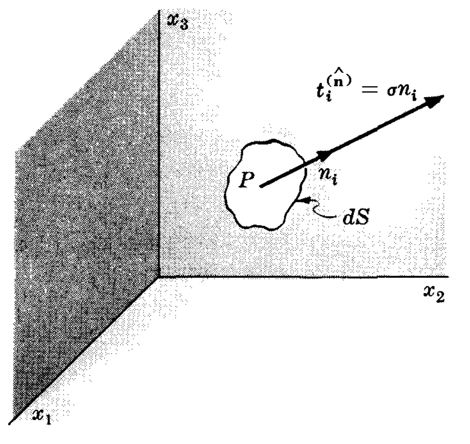


Fig. 2-11

In the three equations (*2.36*), there are four unknowns, namely, the three direction cosines n_i and the principal stress value σ.

For solutions of (*2.36*) other than the trivial one $n_j = 0$, the determinant of coefficients, $|\sigma_{ij} - \delta_{ij}\sigma|$, must vanish. Explicitly,

$$|\sigma_{ij} - \delta_{ij}\sigma| = 0 \quad \text{or} \quad \begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0 \tag{2.37}$$

which upon expansion yields the cubic polynomial in σ,

$$\sigma^3 - \mathrm{I}_\Sigma \sigma^2 + \mathrm{II}_\Sigma \sigma - \mathrm{III}_\Sigma = 0 \tag{2.38}$$

where

$$\mathrm{I}_\Sigma = \sigma_{ii} = \operatorname{tr}\mathbf{\Sigma} \tag{2.39}$$

$$\mathrm{II}_\Sigma = \tfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \tag{2.40}$$

$$\mathrm{III}_\Sigma = |\sigma_{ij}| = \det\mathbf{\Sigma} \tag{2.41}$$

are known respectively as the *first*, *second* and *third stress invariants*.

The three roots of (*2.38*), $\sigma_{(1)}, \sigma_{(2)}, \sigma_{(3)}$ are the three principal stress values. Associated with each principal stress $\sigma_{(k)}$, there is a principal stress direction for which the direction cosines $n_i^{(k)}$ are solutions of the equations

$$(\sigma_{ij} - \sigma_{(k)}\delta_{ij}) n_j^{(k)} = 0 \quad \text{or} \quad (\mathbf{\Sigma} - \sigma_{(k)}\mathbf{I})\cdot\hat{\mathbf{n}}^{(k)} = 0 \quad (k = 1,2,3) \tag{2.42}$$

In (*2.42*) letter subscripts or superscripts enclosed by parentheses are merely labels and as such do not participate in any summation process. The expanded form of (*2.42*) for the *second* principal direction, for example, is therefore

$$\begin{aligned} (\sigma_{11}-\sigma_{(2)})n_1^{(2)} + \sigma_{12}n_2^{(2)} + \sigma_{13}n_3^{(2)} &= 0 \\ \sigma_{21}n_1^{(2)} + (\sigma_{22}-\sigma_{(2)})n_2^{(2)} + \sigma_{23}n_3^{(2)} &= 0 \\ \sigma_{31}n_1^{(2)} + \sigma_{32}n_2^{(2)} + (\sigma_{33}-\sigma_{(2)})n_3^{(2)} &= 0 \end{aligned} \tag{2.43}$$

Because the stress tensor is real and symmetric, the principal stress values are also real.

When referred to principal stress directions, the stress matrix $[\sigma_{ij}]$ is diagonal,

$$[\sigma_{ij}] \equiv \begin{bmatrix} \sigma_{(1)} & 0 & 0 \\ 0 & \sigma_{(2)} & 0 \\ 0 & 0 & \sigma_{(3)} \end{bmatrix} \quad \text{or} \quad [\sigma_{ij}] = \begin{bmatrix} \sigma_{\rm I} & 0 & 0 \\ 0 & \sigma_{\rm II} & 0 \\ 0 & 0 & \sigma_{\rm III} \end{bmatrix} \tag{2.44}$$

in the second form of which Roman numeral subscripts are used to show that the principal stresses are ordered, i.e. $\sigma_{\rm I} > \sigma_{\rm II} > \sigma_{\rm III}$. Since the principal stress directions are coincident with the principal axes of Cauchy's stress quadric, the principal stress values include both the maximum and minimum normal stress components at a point.

In a *principal stress space,* i.e. a space whose axes are in the principal stress directions and whose coordinate unit of measure is stress $(t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})})$ as shown in Fig. 2-12, the arbitrary stress vector $t_i^{(\hat{n})}$ has components

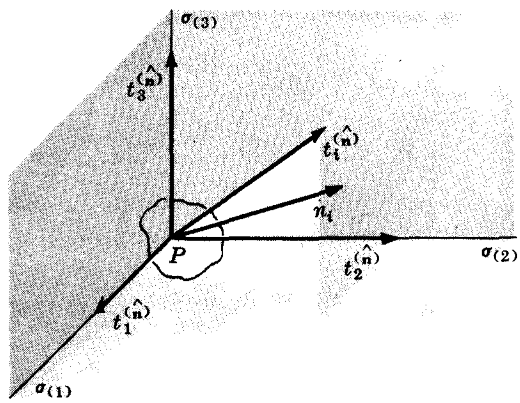


Fig. 2-12

$$t_1^{(\hat{n})} = \sigma_{(1)}n_1, \qquad t_2^{(\hat{n})} = \sigma_{(2)}n_2, \qquad t_3^{(\hat{n})} = \sigma_{(3)}n_3 \tag{2.45}$$

according to (*2.12*). But inasmuch as $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$ for the unit vector n_i, (*2.45*) requires the stress vector $t_i^{(\hat{n})}$ to satisfy the equation

$$\frac{(t_1^{(\hat{n})})^2}{(\sigma_{(1)})^2} + \frac{(t_2^{(\hat{n})})^2}{(\sigma_{(2)})^2} + \frac{(t_3^{(\hat{n})})^2}{(\sigma_{(3)})^2} = 1 \tag{2.46}$$

in stress space. This equation is an ellipsoid known as the *Lamé stress ellipsoid.*

2.11 MAXIMUM AND MINIMUM SHEAR STRESS VALUES

If the stress vector $t_i^{(\hat{n})}$ is resolved into orthogonal components normal and tangential to the surface element dS upon which it acts, the magnitude of the normal component may be determined from (*2.33*) and the magnitude of the tangential or *shearing component* is given by

$$\sigma_S^2 = t_i^{(\hat{n})}t_i^{(\hat{n})} - \sigma_N^2 \tag{2.47}$$

This resolution is shown in Fig. 2-13 where the axes are chosen in the principal stress directions and it is assumed the principal stresses are ordered according to $\sigma_{\mathrm{I}} > \sigma_{\mathrm{II}} > \sigma_{\mathrm{III}}$. Hence from (2.12), the components of $t_i^{(\hat{n})}$ are

$$t_1^{(\hat{n})} = \sigma_{\mathrm{I}} n_1$$
$$t_2^{(\hat{n})} = \sigma_{\mathrm{II}} n_2 \qquad (2.48)$$
$$t_3^{(\hat{n})} = \sigma_{\mathrm{III}} n_3$$

and from (2.33), the normal component magnitude is

$$\sigma_N = \sigma_{\mathrm{I}} n_1^2 + \sigma_{\mathrm{II}} n_2^2 + \sigma_{\mathrm{III}} n_3^2 \qquad (2.49)$$

Substituting (2.48) and (2.49) into (2.47), the squared magnitude of the shear stress as a function of the direction cosines n_i is given by

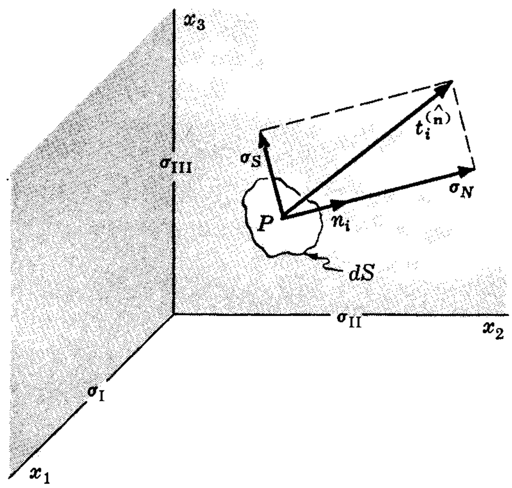


Fig. 2-13

$$\sigma_S^2 = \sigma_{\mathrm{I}}^2 n_1^2 + \sigma_{\mathrm{II}}^2 n_2^2 + \sigma_{\mathrm{III}}^2 n_3^2 - (\sigma_{\mathrm{I}} n_1^2 + \sigma_{\mathrm{II}} n_2^2 + \sigma_{\mathrm{III}} n_3^2)^2 \qquad (2.50)$$

The maximum and minimum values of σ_S may be obtained from (2.50) by the method of *Lagrangian multipliers.* The procedure is to construct the function

$$F = \sigma_S^2 - \lambda n_i n_i \qquad (2.51)$$

in which the scalar λ is called a Lagrangian multiplier. Equation (2.51) is clearly a function of the direction cosines n_i, so that the conditions for stationary (maximum or minimum) values of F are given by $\partial F/\partial n_i = 0$. Setting these partials equal to zero yields the equations

$$n_1\{\sigma_{\mathrm{I}}^2 - 2\sigma_{\mathrm{I}}(\sigma_{\mathrm{I}} n_1^2 + \sigma_{\mathrm{II}} n_2^2 + \sigma_{\mathrm{III}} n_3^2) + \lambda\} = 0 \qquad (2.52a)$$
$$n_2\{\sigma_{\mathrm{II}}^2 - 2\sigma_{\mathrm{II}}(\sigma_{\mathrm{I}} n_1^2 + \sigma_{\mathrm{II}} n_2^2 + \sigma_{\mathrm{III}} n_3^2) + \lambda\} = 0 \qquad (2.52b)$$
$$n_3\{\sigma_{\mathrm{III}}^2 - 2\sigma_{\mathrm{III}}(\sigma_{\mathrm{I}} n_1^2 + \sigma_{\mathrm{II}} n_2^2 + \sigma_{\mathrm{III}} n_3^2) + \lambda\} = 0 \qquad (2.52c)$$

which, together with the condition $n_i n_i = 1$, may be solved for λ and the direction cosines n_1, n_2, n_3, conjugate to the extremum values of shear stress.

One set of solutions to (2.52), and the associated shear stresses from (2.50), are

$$n_1 = \pm 1, \quad n_2 = 0, \quad n_3 = 0; \quad \text{for which } \sigma_S = 0 \qquad (2.53a)$$
$$n_1 = 0, \quad n_2 = \pm 1, \quad n_3 = 0; \quad \text{for which } \sigma_S = 0 \qquad (2.53b)$$
$$n_1 = 0, \quad n_2 = 0, \quad n_3 = \pm 1; \quad \text{for which } \sigma_S = 0 \qquad (2.53c)$$

The shear stress values in (2.53) are obviously minimum values. Furthermore, since (2.35) indicates that shear components vanish on principal planes, the directions given by (2.53) are recognized as principal stress directions.

A second set of solutions to (2.52) may be verified to be given by

$$n_1 = 0, \quad n_2 = \pm 1/\sqrt{2}, \quad n_3 = \pm 1/\sqrt{2}; \quad \text{for which } \sigma_S = (\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})/2 \qquad (2.54a)$$
$$n_1 = \pm 1/\sqrt{2}, \quad n_2 = 0, \quad n_3 = \pm 1/\sqrt{2}; \quad \text{for which } \sigma_S = (\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})/2 \qquad (2.54b)$$
$$n_1 = \pm 1/\sqrt{2}, \quad n_2 = \pm 1/\sqrt{2}, \quad n_3 = 0; \quad \text{for which } \sigma_S = (\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})/2 \qquad (2.54c)$$

Equation (2.54b) gives the maximum shear stress value, which is equal to half the difference of the largest and smallest principal stresses. Also from (2.54b), the maximum shear stress component acts in the plane which bisects the right angle between the directions of the maximum and minimum principal stresses.

2.12 MOHR'S CIRCLES FOR STRESS

A convenient two-dimensional graphical representation of the three-dimensional state of stress at a point is provided by the well-known *Mohr's stress circles.* In developing these, the coordinate axes are again chosen in the principal stress directions at P as shown by Fig. 2-14. The principal stresses are assumed to be distinct and ordered according to

$$\sigma_{\mathrm{I}} > \sigma_{\mathrm{II}} > \sigma_{\mathrm{III}} \tag{2.55}$$

For this arrangement the stress vector $t_i^{(\hat{n})}$ has normal and shear components whose magnitudes satisfy the equations

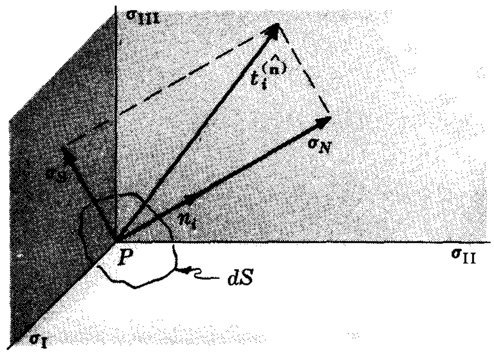


Fig. 2-14

$$\sigma_N = \sigma_{\mathrm{I}} n_1^2 + \sigma_{\mathrm{II}} n_2^2 + \sigma_{\mathrm{III}} n_3^2 \tag{2.56}$$

$$\sigma_N^2 + \sigma_S^2 = \sigma_{\mathrm{I}}^2 n_1^2 + \sigma_{\mathrm{II}}^2 n_2^2 + \sigma_{\mathrm{III}}^2 n_3^2 \tag{2.57}$$

Combining these two expressions with the identity $n_i n_i = 1$ and solving for the direction cosines n_i, results in the equations

$$(n_1)^2 = \frac{(\sigma_N - \sigma_{\mathrm{II}})(\sigma_N - \sigma_{\mathrm{III}}) + (\sigma_S)^2}{(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})(\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}})} \tag{2.58a}$$

$$(n_2)^2 = \frac{(\sigma_N - \sigma_{\mathrm{III}})(\sigma_N - \sigma_{\mathrm{I}}) + (\sigma_S)^2}{(\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})(\sigma_{\mathrm{II}} - \sigma_{\mathrm{I}})} \tag{2.58b}$$

$$(n_3)^2 = \frac{(\sigma_N - \sigma_{\mathrm{I}})(\sigma_N - \sigma_{\mathrm{II}}) + (\sigma_S)^2}{(\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})(\sigma_{\mathrm{III}} - \sigma_{\mathrm{II}})} \tag{2.58c}$$

These equations serve as the basis for Mohr's stress circles, shown in the "stress plane" of Fig. 2-15, for which the σ_N axis is the abscissa, and the σ_S axis is the ordinate.

In (*2.58a*), since $\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}} > 0$ and $\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}} > 0$ from (*2.55*), and since $(n_1)^2$ is non-negative, the numerator of the right-hand side satisfies the relationship

$$(\sigma_N - \sigma_{\mathrm{II}})(\sigma_N - \sigma_{\mathrm{III}}) + (\sigma_S)^2 \geqq 0 \tag{2.59}$$

which represents stress points in the (σ_N, σ_S) plane that are *on* or *exterior* to the circle

$$[\sigma_N - (\sigma_{\mathrm{II}} + \sigma_{\mathrm{III}})/2]^2 + (\sigma_S)^2 = [(\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})/2]^2 \tag{2.60}$$

In Fig. 2-15, this circle is labeled C_1.

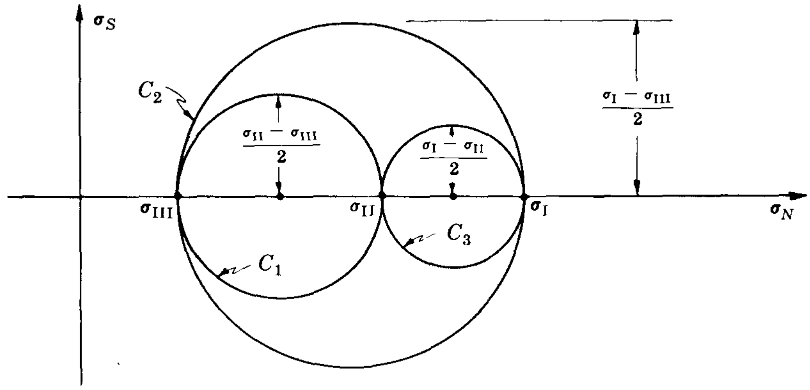


Fig. 2-15

Similarly, for (2.58b), since $\sigma_{II} - \sigma_{III} > 0$ and $\sigma_{II} - \sigma_{I} < 0$ from (2.55), and since $(n_2)^2$ is non-negative, the right hand numerator satisfies

$$(\sigma_N - \sigma_{III})(\sigma_N - \sigma_I) + (\sigma_S)^2 \leqq 0 \qquad (2.61)$$

which represents points *on* or *interior* to the circle

$$[\sigma_N - (\sigma_{III} + \sigma_I)/2]^2 + (\sigma_S)^2 = [(\sigma_{III} - \sigma_I)/2]^2 \qquad (2.62)$$

labeled C_2 in Fig. 2-15. Finally, for (2.58c), since $\sigma_{III} - \sigma_I < 0$ and $\sigma_{III} - \sigma_{II} < 0$ from (2.55), and since $(n_3)^2$ is non-negative,

$$(\sigma_N - \sigma_I)(\sigma_N - \sigma_{II}) + (\sigma_S)^2 \geqq 0 \qquad (2.63)$$

which represents points *on* or *exterior* to the circle

$$[\sigma_N - (\sigma_I + \sigma_{II})/2]^2 + (\sigma_S)^2 = [(\sigma_I - \sigma_{II})/2]^2 \qquad (2.64)$$

labeled C_3 in Fig. 2-15.

Since each "stress point" (pair of values of σ_N and σ_S) in the (σ_N, σ_S) plane represents a particular stress vector $t_i^{(\hat{n})}$, the state of stress at P expressed by (2.58) is represented in Fig. 2-15 as the shaded area bounded by the Mohr's stress circles. The diagram confirms a maximum shear stress of $(\sigma_I - \sigma_{III})/2$ as was determined analytically in Section 2.11. Frequently, because the sign of the shear stress is not of critical importance, only the top half of this symmetrical diagram is drawn.

The relationship between Mohr's stress diagram and the physical state of stress may be established through consideration of Fig. 2-16, which shows the first octant of a sphere of the continuum centered at point P. The normal n_i at the arbitrary point Q of the spherical surface ABC simulates the normal to the surface element dS at point P. Because of the symmetry properties of the stress tensor and the fact that principal stress axes are used in Fig. 2-16, the state of stress at P is completely represented through the totality of locations Q can occupy on the surface ABC. In the figure, circle arcs KD, GE and FH designate locations for Q along which one direction cosine of n_i has a constant value. Specifically,

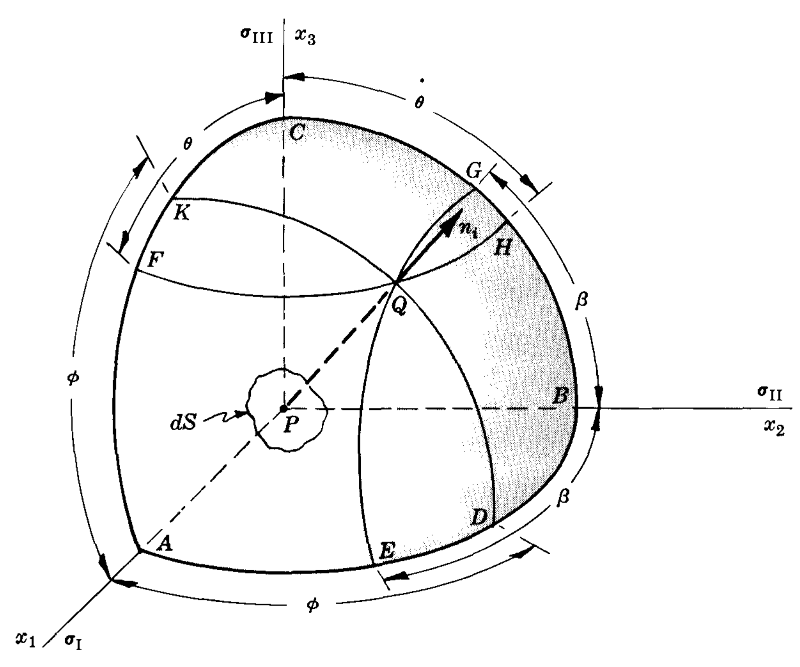


Fig. 2-16

$$n_1 = \cos\phi \text{ on } KD, \qquad n_2 = \cos\beta \text{ on } GE, \qquad n_3 = \cos\theta \text{ on } FH$$

and, on the bounding circle arcs BC, CA and AB,

$$n_1 = \cos\pi/2 = 0 \text{ on } BC, \qquad n_2 = \cos\pi/2 = 0 \text{ on } CA, \qquad n_3 = \cos\pi/2 = 0 \text{ on } AB$$

According to the first of these and the equation (*2.58a*), stress vectors for Q located on BC will have components given by stress points on the circle C_1 in Fig. 2-15. Likewise, CA in Fig. 2-16 corresponds to the circle C_2, and AB to the circle C_3 in Fig. 2-15.

The stress vector components σ_N and σ_S for an arbitrary location of Q may be determined by the construction shown in Fig. 2-17. Thus point e may be located on C_3 by drawing the radial line from the center of C_3 at the angle 2β. Note that angles in the physical space of Fig. 2-16 are doubled in the stress space of Fig. 2-17 (arc AB subtends 90° in Fig. 2-16 whereas the conjugate stress points σ_I and σ_{II} are 180° apart on C_3). In the same way, points g, h and f are located in Fig. 2-17 and the appropriate pairs joined by circle arcs having their centers on the σ_N axis. The intersection of circle arcs ge and hf represents the components σ_N and σ_S of the stress vector $t_i^{(\hat{n})}$ on the plane having the normal direction n_i at Q in Fig. 2-16.

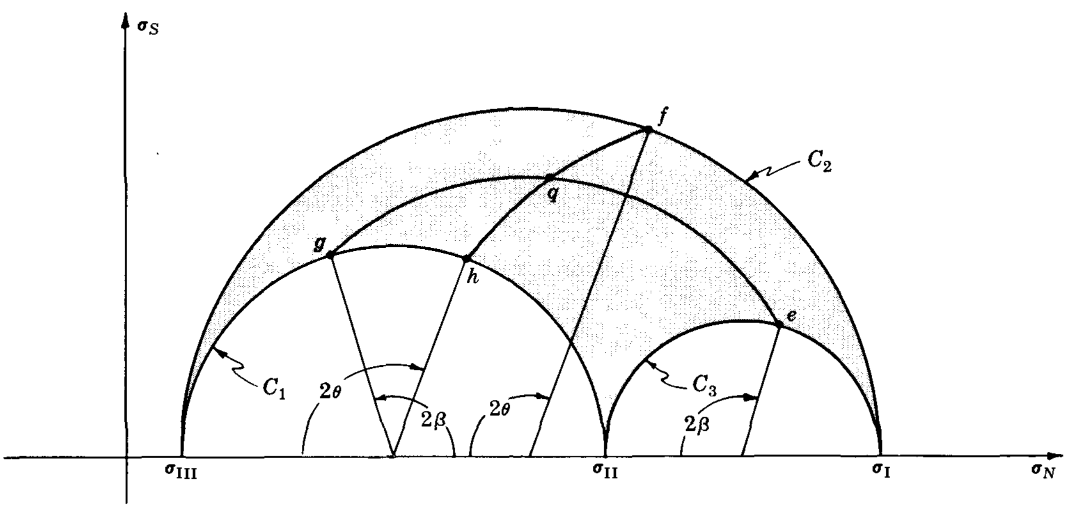


Fig. 2-17

2.13 PLANE STRESS

In the case where one and only one of the principal stresses is zero a state of *plane stress* is said to exist. Such a situation occurs at an unloaded point on the free surface bounding a body. If the principal stresses are ordered, the Mohr's stress circles will have one of the characterizations appearing in Fig. 2-18.

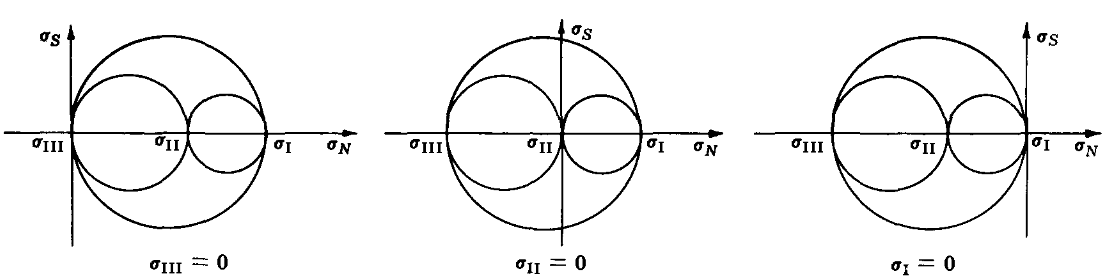


Fig. 2-18

If the principal stresses are not ordered and the direction of the zero principal stress is taken as the x_3 direction, the state of stress is termed plane stress parallel to the x_1x_2 plane. For arbitrary choice of orientation of the orthogonal axes x_1 and x_2 in this case, the stress matrix has the form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.65}$$

The stress quadric for this plane stress is a cylinder with its base lying in the x_1x_2 plane and having the equation

$$\sigma_{11}x_1^2 + 2\sigma_{12}x_1x_2 + \sigma_{22}x_2^2 = \pm k^2 \tag{2.66}$$

Frequently in elementary books on Strength of Materials a state of plane stress is represented by a single Mohr's circle. As seen from Fig. 2-18 this representation is necessarily incomplete since all three circles are required to show the complete stress picture. In particular, the maximum shear stress value at a point will not be given if the single circle presented happens to be one of the inner circles of Fig. 2-18. A single circle Mohr's diagram is able, however, to display the stress points for all those planes at the point P which include the zero principal stress axis. For such planes, if the coordinate axes are chosen in accordance with the stress representation given in (*2.65*), the single plane stress Mohr's circle has the equation

$$[\sigma_N - (\sigma_{11} + \sigma_{22})/2]^2 + (\sigma_S)^2 = [(\sigma_{11} - \sigma_{22})/2]^2 + (\sigma_{12})^2 \tag{2.67}$$

The essential features in the construction of this circle are illustrated in Fig. 2-19. The circle is drawn by locating the center C at $\sigma_N = (\sigma_{11} + \sigma_{22})/2$ and using the radius $R = \sqrt{[(\sigma_{11} - \sigma_{22})/2]^2 + (\sigma_{12})^2}$ given in (*2.67*). Point A on the circle represents the stress state on the surface element whose normal is n_1 (the right-hand face of the rectangular parallelepiped shown in Fig. 2-19). Point B on the circle represents the stress state on the top surface of the parallelepiped with normal n_2. Principal stress points σ_{I} and σ_{II} are so labeled, and points E and D on the circle are points of maximum shear stress value.

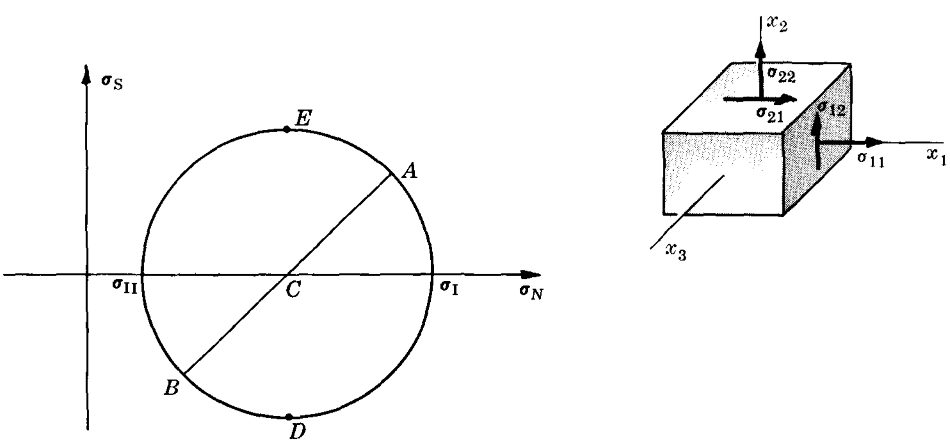


Fig. 2-19

2.14 DEVIATOR AND SPHERICAL STRESS TENSORS

It is very often useful to split the stress tensor σ_{ij} into two component tensors, one of which (the *spherical* or *hydrostatic stress tensor*) has the form

$$\mathbf{\Sigma}_M = \sigma_M \mathbf{I} = \begin{pmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{pmatrix} \tag{2.68}$$

where $\sigma_M = -p = \sigma_{kk}/3$ is the mean normal stress, and the second (the *deviator stress tensor*) has the form

$$\mathbf{\Sigma}_D = \begin{pmatrix} \sigma_{11}-\sigma_M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_M \end{pmatrix} \equiv \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \tag{2.69}$$

This decomposition is expressed by the equations

$$\sigma_{ij} = \delta_{ij}\sigma_{kk}/3 + s_{ij} \quad \text{or} \quad \mathbf{\Sigma} = \sigma_M \mathbf{I} + \mathbf{\Sigma}_D \tag{2.70}$$

The principal directions of the deviator stress tensor s_{ij} are the same as those of the stress tensor σ_{ij}. Thus *principal deviator stress* values are

$$s_{(k)} = \sigma_{(k)} - \sigma_M \tag{2.71}$$

The characteristic equation for the deviator stress tensor, comparable to (*2.38*) for the stress tensor, is the cubic

$$s^3 + \mathrm{II}_{\Sigma_D} s - \mathrm{III}_{\Sigma_D} = 0 \quad \text{or} \quad s^3 + (s_I s_{II} + s_{II} s_{III} + s_{III} s_I)s - s_I s_{II} s_{III} = 0 \tag{2.72}$$

It is easily shown that the first invariant of the deviator stress tensor I_{Σ_D} is identically zero, which accounts for its absence in (*2.72*).

Solved Problems

STATE OF STRESS AT A POINT. STRESS VECTOR. STRESS TENSOR (Sec. 2.1-2.6)

2.1. At the point P the stress vectors $t_i^{(\hat{n})}$ and $t_i^{(\hat{n}^*)}$ act on the respective surface elements $n_i \Delta S$ and $n_i^* \Delta S^*$. Show that the component of $t_i^{(\hat{n})}$ in the direction of n_i^* is equal to the component of $t_i^{(\hat{n}^*)}$ in the direction of n_i.

It is required to show that

$$t_i^{(\hat{n}^*)} n_i = t_i^{(\hat{n})} n_i^*$$

From (*2.12*) $t_i^{(\hat{n}^*)} n_i = \sigma_{ji} n_j^* n_i$, and by (*2.22*) $\sigma_{ji} = \sigma_{ij}$, so that

$$\sigma_{ji} n_j^* n_i = (\sigma_{ij} n_i) n_j^* = t_j^{(\hat{n})} n_j^*$$

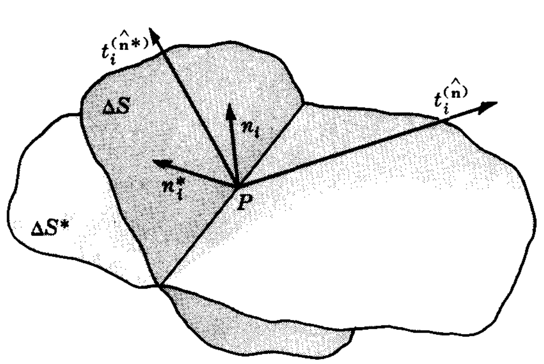


Fig. 2-20

2.2. The stress tensor values at a point P are given by the array

$$\boldsymbol{\Sigma} = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Determine the traction (stress) vector on the plane at P whose unit normal is $\hat{\mathbf{n}} = (2/3)\hat{\mathbf{e}}_1 - (2/3)\hat{\mathbf{e}}_2 + (1/3)\hat{\mathbf{e}}_3$.

From (*2.12*), $\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \boldsymbol{\Sigma}$. The multiplication is best carried out in the matrix form of (*2.13*):

$$[t_1^{(\hat{\mathbf{n}})}, t_2^{(\hat{\mathbf{n}})}, t_3^{(\hat{\mathbf{n}})}] = [2/3, -2/3, 1/3]\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} = \left[\frac{14}{3} - \frac{2}{3}, \frac{-10}{3}, \frac{-4}{3} + \frac{4}{3}\right]$$

Thus $\mathbf{t}^{(\hat{\mathbf{n}})} = 4\hat{\mathbf{e}}_1 - \frac{10}{3}\hat{\mathbf{e}}_2$.

2.3. For the traction vector of Problem 2.2, determine (*a*) the component perpendicular to the plane, (*b*) the magnitude of $t_i^{(\hat{\mathbf{n}})}$, (*c*) the angle between $t_i^{(\hat{\mathbf{n}})}$ and $\hat{\mathbf{n}}$.

(*a*) $t_i^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = (4\hat{\mathbf{e}}_1 - \frac{10}{3}\hat{\mathbf{e}}_2) \cdot (\frac{2}{3}\hat{\mathbf{e}}_1 - \frac{2}{3}\hat{\mathbf{e}}_2 + \frac{1}{3}\hat{\mathbf{e}}_3) = 44/9$

(*b*) $|t_i^{(\hat{\mathbf{n}})}| = \sqrt{16 + 100/9} = 5.2$

(*c*) Since $t_i^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = |t_i^{(\hat{\mathbf{n}})}| \cos\theta$, $\cos\theta = (44/9)/5.2 = 0.94$ and $\theta = 20°$.

2.4. The stress vectors acting on the three coordinate planes are given by $t_i^{(\hat{\mathbf{e}}_1)}$, $t_i^{(\hat{\mathbf{e}}_2)}$ and $t_i^{(\hat{\mathbf{e}}_3)}$. Show that the sum of the squares of the magnitudes of these vectors is independent of the orientation of the coordinate planes.

Let S be the sum in question. Then

$$S = t_i^{(\hat{\mathbf{e}}_1)} t_i^{(\hat{\mathbf{e}}_1)} + t_i^{(\hat{\mathbf{e}}_2)} t_i^{(\hat{\mathbf{e}}_2)} + t_i^{(\hat{\mathbf{e}}_3)} t_i^{(\hat{\mathbf{e}}_3)}$$

which from (*2.7*) becomes $S = \sigma_{1i}\sigma_{1i} + \sigma_{2i}\sigma_{2i} + \sigma_{3i}\sigma_{3i} = \sigma_{ji}\sigma_{ji}$, an invariant.

2.5. The state of stress at a point is given by the stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{pmatrix}$$

where a, b, c are constants and σ is some stress value. Determine the constants a, b and c so that the stress vector on the *octahedral* plane $(\hat{\mathbf{n}} = (1/\sqrt{3})\hat{\mathbf{e}}_1 + (1/\sqrt{3})\hat{\mathbf{e}}_2 + (1/\sqrt{3})\hat{\mathbf{e}}_3)$ vanishes.

In matrix form, $t_i^{(\hat{\mathbf{n}})} = \sigma_{ij}n_j$ must be zero for the given stress tensor and normal vector.

$$\begin{bmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{hence} \qquad \begin{aligned} a + b &= -1 \\ a + c &= -1 \\ b + c &= -1 \end{aligned}$$

Solving these equations, $a = b = c = -1/2$. Therefore the solution tensor is

$$\sigma_{ij} = \begin{pmatrix} \sigma & -\sigma/2 & -\sigma/2 \\ -\sigma/2 & \sigma & -\sigma/2 \\ -\sigma/2 & -\sigma/2 & \sigma \end{pmatrix}$$

2.6. The stress tensor at point P is given by the array

$$\boldsymbol{\Sigma} = \begin{pmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Determine the stress vector on the plane passing through P and parallel to the plane ABC shown in Fig. 2-21.

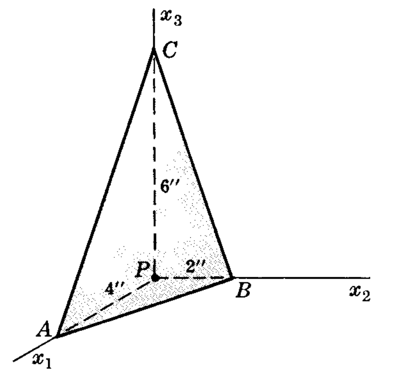


Fig. 2-21

The equation of the plane ABC is $3x_1 + 6x_2 + 2x_3 = 12$, and the unit normal to the plane is therefore (see Problem 1.2)

$$\hat{\mathbf{n}} = \tfrac{3}{7}\hat{\mathbf{e}}_1 + \tfrac{6}{7}\hat{\mathbf{e}}_2 + \tfrac{2}{7}\hat{\mathbf{e}}_3$$

From (*2.14*), the stress vector may be determined by matrix multiplication,

$$[3/7,\ 6/7,\ 2/7]\begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \frac{1}{7}[-9,\ 5,\ 10]$$

Thus $\quad \mathbf{t}^{(\hat{\mathbf{n}})} = \dfrac{-9}{7}\hat{\mathbf{e}}_1 + \dfrac{5}{7}\hat{\mathbf{e}}_2 + \dfrac{10}{7}\hat{\mathbf{e}}_3.$

2.7. The state of stress throughout a continuum is given with respect to the Cartesian axes $Ox_1x_2x_3$ by the array

$$\boldsymbol{\Sigma} = \begin{pmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{pmatrix}$$

Determine the stress vector acting at the point $P(2, 1, \sqrt{3})$ of the plane that is tangent to the cylindrical surface $x_2^2 + x_3^2 = 4$ at P.

At P the stress components are given by

$$\boldsymbol{\Sigma} = \begin{pmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{pmatrix}$$

The unit normal to the surface at P is determined from $\operatorname{grad}\phi = \nabla\phi = \nabla(x_2^2 + x_3^2 - 4)$. Thus $\nabla\phi = 2x_2\hat{\mathbf{e}}_2 + 2x_3\hat{\mathbf{e}}_3$ and so

$$\nabla\phi = 2\hat{\mathbf{e}}_2 + 2\sqrt{3}\,\hat{\mathbf{e}}_3 \quad \text{at } P$$

Therefore the unit normal at P is $\hat{\mathbf{n}} = \dfrac{\hat{\mathbf{e}}_2}{2} + \dfrac{\sqrt{3}}{2}\hat{\mathbf{e}}_3$. This may also be seen in Fig. 2-22. Finally the stress vector at P on the plane $\perp$ to $\hat{\mathbf{n}}$ is given by

$$\begin{bmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3 \\ \sqrt{3} \end{bmatrix}$$

or $\quad \mathbf{t}^{(\hat{\mathbf{n}})} = 5\hat{\mathbf{e}}_1/2 + 3\hat{\mathbf{e}}_2 + \sqrt{3}\,\hat{\mathbf{e}}_3.$

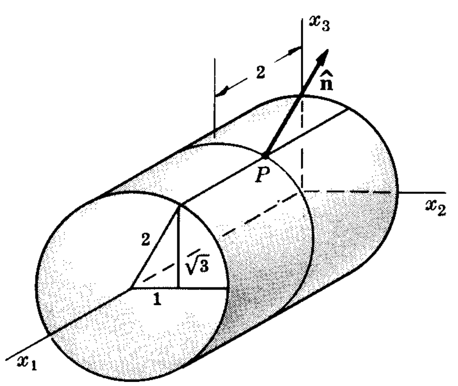


Fig. 2-22

EQUILIBRIUM EQUATIONS (Sec. 2.7)

2.8. For the distribution of the state of stress given in Problem 2.7, what form must the body force components have if the equilibrium equations (*2.24*) are to be satisfied everywhere.

Computing (*2.24*) directly from $\mathbf{\Sigma}$ given in Problem 2.7,

$$\begin{aligned} 3x_2 + 10x_2 + 0 + \rho b_1 &= 0 \\ 0 + 0 + 2 + \rho b_2 &= 0 \\ 0 + 0 + 0 + \rho b_3 &= 0 \end{aligned}$$

These equations are satisfied when $b_1 = -13x_2/\rho$, $b_2 = -2/\rho$, $b_3 = 0$.

2.9. Derive (*2.20*) from (*2.19*), page 49.

Starting with equation (*2.19*),

$$\int_S \epsilon_{ijk} x_j t_k^{(\hat{n})}\, dS + \int_V \epsilon_{ijk} x_j \rho b_k\, dV = 0$$

substitute $t_i^{(\hat{n})} = \sigma_{ji} n_j$ in the surface integral and convert the result to a volume integral by (*1.157*):

$$\int_S (\epsilon_{ijk} x_j \sigma_{pk}) n_p\, dS = \int_V (\epsilon_{ijk} x_j \sigma_{pk})_{,p}\, dV$$

Carrying out the indicated differentiation in this volume integral and combining with the first volume integral gives

$$\int_V \epsilon_{ijk}\{x_{j,p}\sigma_{pk} + x_j(\sigma_{pk,p} + \rho b_k)\}\, dV = 0$$

But from equilibrium equations, $\sigma_{pk,p} + \rho b_k = 0$; and since $x_{j,p} = \delta_{jp}$, this volume integral reduces to (*2.20*), $\int_V \epsilon_{ijk}\sigma_{jk}\, dV = 0$.

STRESS TRANSFORMATIONS (Sec. 2.8)

2.10. The state of stress at a point is given with respect to the Cartesian axes $Ox_1x_2x_3$ by the array

$$\mathbf{\Sigma} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

Determine the stress tensor $\mathbf{\Sigma}'$ for the rotated axes $Ox_1'x_2'x_3'$ related to the unprimed axes by the transformation tensor

$$\mathbf{A} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & -1/2 \\ -1/\sqrt{2} & 1/2 & -1/2 \end{pmatrix}$$

The stress transformation law is given by (*2.27*) as $\sigma'_{ij} = a_{ip}a_{jq}\sigma_{pq}$ or $\mathbf{\Sigma}' = \mathbf{A}\cdot\mathbf{\Sigma}\cdot\mathbf{A}_c$. The detailed calculation is best carried out by the matrix multiplication $[\sigma'_{ij}] = [a_{ip}][\sigma_{pq}][a_{qj}]$ given in (*2.29*). Thus

$$[\sigma'_{ij}] = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & -1/2 \\ -1/\sqrt{2} & 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 2 & -2 & 0 \\ -2 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & 1/2 \\ 1/\sqrt{2} & -1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1-\sqrt{2} & -1 \\ 2 & -1 & 1+\sqrt{2} \end{bmatrix}$$

2.11. Show that the stress transformation law may be derived from (*2.33*), the equation $\sigma_N = \sigma_{ij} n_i n_j$ expressing the normal stress value on an arbitrary plane having the unit normal vector n_i.

Since σ_N is a zero-order tensor, it is given with respect to an arbitrary set of primed or unprimed axes in the same form as

$$\sigma_N = \sigma'_{ij} n'_i n'_j = \sigma_{ij} n_i n_j$$

and since by (*1.94*) $n'_i = a_{ij} n_j$,

$$\sigma'_{ij} n'_i n'_j = \sigma'_{ij} a_{ip} n_p a_{jq} n_q = \sigma_N = \sigma_{pq} n_p n_q$$

where new dummy indices have been introduced in the last term. Therefore

$$(\sigma'_{ij} a_{ip} a_{jq} - \sigma_{pq}) n_p n_q = 0$$

and since the directions of the unprimed axes are arbitrary,

$$\sigma'_{ij} a_{ip} a_{jq} = \sigma_{pq}$$

2.12. For the unprimed axes in Fig. 2-23, the stress tensor is given by

$$\sigma_{ij} = \begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{pmatrix}$$

Determine the stress tensor for the primed axes specified as shown in the figure.

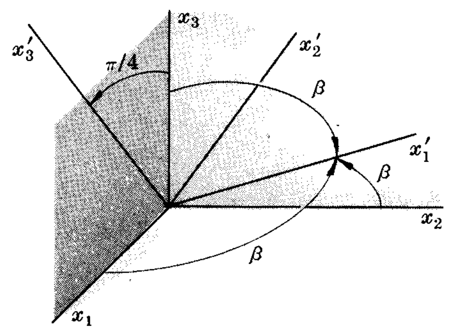


Fig. 2-23

It is first necessary to determine completely the transformation matrix **A**. Since x'_1 makes equal angles with the x_i axes, the first row of the transformation table together with a_{33} is known. Thus

	x_1	x_2	x_3
x'_1	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$
x'_2			
x'_3			$1/\sqrt{2}$

Using the orthogonality equations $a_{ij} a_{ik} = \delta_{jk}$, the transformation matrix is determined by computing the missing entries in the table. It is left as an exercise for the student to show that

$$[a_{ij}] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Therefore

$$[\sigma'_{ij}] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \tau/\sqrt{3} & \tau/\sqrt{3} & \tau/\sqrt{3} \\ -2\tau/\sqrt{6} & \tau/\sqrt{6} & \tau/\sqrt{6} \\ 0 & -\tau/\sqrt{2} & \tau/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{bmatrix}$$

The result obtained here is not surprising when one considers the Mohr's circles for the state of stress having three equal principal stress values.

CAUCHY'S STRESS QUADRIC (Sec. 2.9)

2.13. Determine the Cauchy stress quadric at P for the following states of stress:

(*a*) uniform tension $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma;\ \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$

(*b*) uniaxial tension $\sigma_{11} = \sigma;\ \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$

(*c*) simple shear $\sigma_{12} = \sigma_{21} = \tau;\ \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$

(*d*) plane stress with $\sigma_{11} = \sigma = \sigma_{22};\ \sigma_{12} = \sigma_{21} = \tau;\ \sigma_{33} = \sigma_{31} = \sigma_{23} = 0.$

From *(2.32)*, the quadric surface is given in symbolic notation by the equation $\boldsymbol{\zeta} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{\zeta} = \pm k^2$. Thus using the matrix form,

(*a*) $$[\zeta_1, \zeta_2, \zeta_3] \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \sigma\zeta_1^2 + \sigma\zeta_2^2 + \sigma\zeta_3^2 = \pm k^2$$

Hence the quadric surface for uniform tension is the sphere $\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = \pm k^2/\sigma$.

(*b*) $$[\zeta_1, \zeta_2, \zeta_3] \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \sigma\zeta_1^2 = \pm k^2$$

Hence the quadric surface for uniaxial tension is a circular cylinder along the tension axis.

(*c*) $$[\zeta_1, \zeta_2, \zeta_3] \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = 2\tau\zeta_1\zeta_2 = \pm k^2$$

and so the quadric surface for simple shear is a hyperbolic cylinder parallel to the ζ_3 axis.

(*d*) $$[\zeta_1, \zeta_2, \zeta_3] \begin{bmatrix} \sigma & \tau & 0 \\ \tau & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \sigma\zeta_1^2 + 2\tau\zeta_1\zeta_2 + \sigma\zeta_2^2 = \pm k^2$$

and so the quadric surface for plane stress is a general conic cylinder parallel to the zero principal axis.

2.14. Show that the Cauchy stress quadric for a state of stress represented by

$$\boldsymbol{\Sigma} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

is an ellipsoid (the stress ellipsoid) when a, b and c are all of the same sign.

The equation of the quadric is given by

$$[\zeta_1, \zeta_2, \zeta_3] \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = a\zeta_1^2 + b\zeta_2^2 + c\zeta_3^2 = \pm k^2$$

Therefore the quadric surface is the ellipsoid $\dfrac{\zeta_1^2}{bc} + \dfrac{\zeta_2^2}{ac} + \dfrac{\zeta_3^2}{ab} = \dfrac{\pm k^2}{abc}$.

PRINCIPAL STRESSES (Sec. 2.10-2.11)

2.15. The stress tensor at a point P is given with respect to the axes $Ox_1x_2x_3$ by the values

$$\sigma_{ij} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

Determine the principal stress values and the principal stress directions represented by the axes $Ox_1^*x_2^*x_3^*$.

From (*2.37*) the principal stress values σ are given by

$$\begin{vmatrix} 3-\sigma & 1 & 1 \\ 1 & -\sigma & 2 \\ 1 & 2 & -\sigma \end{vmatrix} = 0 \quad \text{or, upon expansion,} \quad (\sigma+2)(\sigma-4)(\sigma-1) = 0$$

The roots are the principal stress values $\sigma_{(1)} = -2$, $\sigma_{(2)} = 1$, $\sigma_{(3)} = 4$. Let the x_1^* axis be the direction of $\sigma_{(1)}$, and let $n_i^{(1)}$ be the direction cosines of this axis. Then from (*2.42*),

$$\begin{aligned} (3+2)n_1^{(1)} + n_2^{(1)} + n_3^{(1)} &= 0 \\ n_1^{(1)} + 2n_2^{(1)} + 2n_3^{(1)} &= 0 \\ n_1^{(1)} + 2n_2^{(1)} + 2n_3^{(1)} &= 0 \end{aligned}$$

Hence $n_1^{(1)} = 0$; $n_2^{(1)} = -n_3^{(1)}$ and since $n_in_i = 1$, $(n_2^{(1)})^2 = 1/2$. Therefore $n_1^{(1)} = 0$, $n_2^{(1)} = 1/\sqrt{2}$, $n_3^{(1)} = -1/\sqrt{2}$.

Likewise, let x_2^* be associated with $\sigma_{(2)}$. Then from (*2.42*),

$$\begin{aligned} 2n_1^{(2)} + n_2^{(2)} + n_3^{(2)} &= 0 \\ n_1^{(2)} - n_2^{(2)} + 2n_3^{(2)} &= 0 \\ n_1^{(2)} + 2n_2^{(2)} - n_3^{(2)} &= 0 \end{aligned}$$

so that $n_1^{(2)} = 1/\sqrt{3}$, $n_2^{(2)} = -1/\sqrt{3}$, $n_3^{(2)} = -1/\sqrt{3}$.

Finally, let x_3^* be associated with $\sigma_{(3)}$. Then from (*2.42*),

$$\begin{aligned} -n_1^{(3)} + n_2^{(3)} + n_3^{(3)} &= 0 \\ n_1^{(3)} - 4n_2^{(3)} + 2n_3^{(3)} &= 0 \\ n_1^{(3)} + 2n_2^{(3)} - 4n_3^{(3)} &= 0 \end{aligned}$$

so that $n_1^{(3)} = -2/\sqrt{6}$, $n_2^{(3)} = -1/\sqrt{6}$, $n_3^{(3)} = -1/\sqrt{6}$.

2.16. Show that the transformation tensor of direction cosines determined in Problem 2.15 transforms the original stress tensor into the diagonal principal axes stress tensor.

According to (*2.29*), $[\sigma_{ij}^*] = [a_{ip}][\sigma_{pq}][a_{qj}]$, which for the problem at hand becomes

$$[\sigma_{ij}^*] = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ -2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sqrt{2} & \sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ -8/\sqrt{6} & -4/\sqrt{6} & -4/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

2.17. Determine the principal stress values and principal directions for the stress tensor

$$\sigma_{ij} = \begin{pmatrix} \tau & \tau & \tau \\ \tau & \tau & \tau \\ \tau & \tau & \tau \end{pmatrix}$$

From *(2.37)*, $\begin{vmatrix} \tau-\sigma & \tau & \tau \\ \tau & \tau-\sigma & \tau \\ \tau & \tau & \tau-\sigma \end{vmatrix} = 0$ or $(\tau-\sigma)[-2\tau\sigma+\sigma^2]+2\tau^2\sigma = [3\tau-\sigma]\sigma^2 = 0$.

Hence $\sigma_{(1)} = 0$, $\sigma_{(2)} = 0$, $\sigma_{(3)} = 3\tau$. For $\sigma_{(3)} = 3\tau$, *(2.42)* yield

$$-2n_1^{(3)} + n_2^{(3)} + n_3^{(3)} = 0, \quad n_1^{(3)} - 2n_2^{(3)} + n_3^{(3)} = 0, \quad n_1^{(3)} + n_2^{(3)} - 2n_3^{(3)} = 0$$

and therefore $n_1^{(3)} = n_2^{(3)} = n_3^{(3)} = 1/\sqrt{3}$. For $\sigma_{(1)} = \sigma_{(2)} = 0$, *(2.42)* yield

$$n_1 + n_2 + n_3 = 0, \quad n_1 + n_2 + n_3 = 0, \quad n_1 + n_2 + n_3 = 0$$

which together with $n_i n_i = 1$ are insufficient to determine uniquely the first and second principal directions. Thus any pair of axes perpendicular to the $n_i^{(3)}$ direction and perpendicular to each other may serve as principal axes. For example, consider the axes determined in Problem 2.12, for which the transformation matrix is

$$[a_{ij}] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

According to the transformation law *(2.29)*, the principal stress matrix $[\sigma_{ij}^*]$ is given by

$$[\sigma_{ij}^*] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \tau & \tau & \tau \\ \tau & \tau & \tau \\ \tau & \tau & \tau \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}\,\tau & \sqrt{3}\,\tau & \sqrt{3}\,\tau \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3\tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.18. Show that the axes $Ox_1^* x_2^* x_3^*$, (where x_2^*, x_3 and x_3^* are in the same vertical plane, and x_1^*, x_1 and x_2 are in the same horizontal plane) are also principal axes for the stress tensor of Problem 2.17.

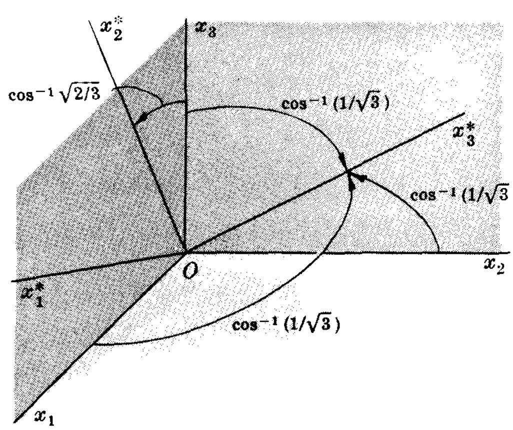


Fig. 2-24

The transformation matrix $[a_{ij}]$ relating the two sets of axes clearly has the known elements

$$[a_{ij}] = \begin{bmatrix} - & - & 0 \\ - & - & \sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

as is evident from Fig. 2-24. From the orthogonality conditions $a_{ij}a_{ik} = \delta_{jk}$, the remaining four elements are determined so that

$$[a_{ij}] \quad = \quad \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

As before,

$$[\sigma^*_{ij}] \quad = \quad \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \tau & \tau & \tau \\ \tau & \tau & \tau \\ \tau & \tau & \tau \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{3}\,\tau & \sqrt{3}\,\tau & \sqrt{3}\,\tau \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \quad = \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\tau \end{bmatrix}$$

2.19. Show that the principal stresses $\sigma^*_{(k)}$ and the stress components σ_{ij} for an arbitrary set of axes referred to the principal directions through the transformation coefficients a_{ij} are related by $\sigma_{ij} = \sum_{p=1}^{3} a_{pi}a_{pj}\sigma_p$.

From the transformation law for stress *(2.27)*, $\sigma_{ij} = a_{pi}a_{qj}\sigma^*_{pq}$; but since σ^*_{pq} are principal stresses, there are only three terms on the right side of this equation, and in each $p = q$. Therefore the right hand side may be written in form $\sigma_{ij} = \sum_{p=1}^{3} a_{pi}a_{pj}\sigma^*_p$.

2.20. Prove that $\sigma_{ij}\sigma_{ik}\sigma_{kj}$ is an invariant of the stress tensor.

By the transformation law *(2.27)*,

$$\begin{aligned} \sigma'_{ij}\sigma'_{ik}\sigma'_{kj} &= a_{ip}a_{jq}\sigma_{pq}a_{ir}a_{ks}\sigma_{rs}a_{km}a_{jn}\sigma_{mn} \\ &= (a_{ip}a_{ir})(a_{jq}a_{jn})(a_{ks}a_{km})\sigma_{pq}\sigma_{rs}\sigma_{mn} \\ &= \delta_{pr}\delta_{qn}\delta_{sm}\sigma_{pq}\sigma_{rs}\sigma_{mn} \\ &= (\delta_{pr}\sigma_{pq})(\delta_{qn}\sigma_{mn})(\delta_{sm}\sigma_{rs}) \\ &= \sigma_{rq}\sigma_{qm}\sigma_{rm} \;=\; \sigma_{ij}\sigma_{ik}\sigma_{kj} \end{aligned}$$

2.21. Evaluate directly the invariants $I_\Sigma, II_\Sigma, III_\Sigma$ for the stress tensor

$$\sigma_{ij} \quad = \quad \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Determine the principal stress values for this state of stress and show that the diagonal form of the stress tensor yields the same values for the stress invariants.

From *(2.39)*, $I_\Sigma = \sigma_{ii} = 6 + 6 + 8 = 20.$

From *(2.40)*,
$$\begin{aligned} II_\Sigma &= (1/2)(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \\ &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}\sigma_{12} - \sigma_{23}\sigma_{23} - \sigma_{31}\sigma_{31} \\ &= 36 + 48 + 48 - 9 = 123. \end{aligned}$$

From *(2.41)*, $III_\Sigma = |\sigma_{ij}| = 6(48) + 3(-24) = 216.$

The principal stress values of σ_{ij} are $\sigma_I = 3$, $\sigma_{II} = 8$, $\sigma_{III} = 9$. In terms of principal values,

$$\begin{aligned} I_\Sigma &= \sigma_I + \sigma_{II} + \sigma_{III} = 3 + 8 + 9 = 20 \\ II_\Sigma &= \sigma_I\sigma_{II} + \sigma_{II}\sigma_{III} + \sigma_{III}\sigma_I = 24 + 72 + 27 = 123 \\ III_\Sigma &= \sigma_I\sigma_{II}\sigma_{III} = (24)9 = 216 \end{aligned}$$

2.22. The *octahedral plane* is the plane which makes equal angles with the principal stress directions. Show that the shear stress on this plane, the so-called *octahedral shear stress*, is given by

$$\sigma_{OCT} = \tfrac{1}{3}\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}$$

With respect to the principal axes, the normal to the octahedral plane is given by

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$$

Hence from (*2.12*) the stress vector on the octahedral plane is

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3) \cdot (\sigma_I\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \sigma_{II}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \sigma_{III}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3) = \frac{1}{\sqrt{3}}(\sigma_I\hat{\mathbf{e}}_1 + \sigma_{II}\hat{\mathbf{e}}_2 + \sigma_{III}\hat{\mathbf{e}}_3)$$

and its normal component is

$$\sigma_N = \hat{\mathbf{n}} \cdot \mathbf{t}^{(\hat{\mathbf{n}})} = \tfrac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III})$$

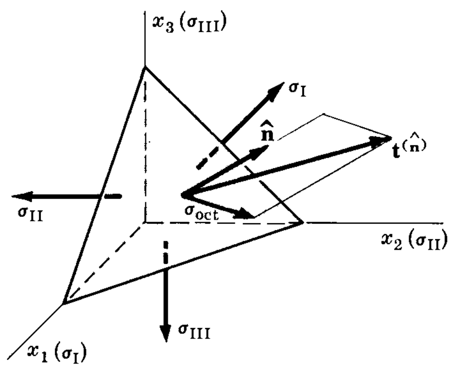


Fig. 2-25

Therefore the shear component is

$$\begin{aligned}\sigma_{OCT} &= \sqrt{\mathbf{t}^{(\hat{\mathbf{n}})} \cdot \mathbf{t}^{(\hat{\mathbf{n}})} - \sigma_N^2} = \{\tfrac{1}{3}(\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2) - \tfrac{1}{9}(\sigma_I + \sigma_{II} + \sigma_{III})^2\}^{1/2} \\ &= \tfrac{1}{3}\{3(\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2) - (\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2 + 2\sigma_I\sigma_{II} + 2\sigma_{II}\sigma_{III} + 2\sigma_{III}\sigma_I)\}^{1/2} \\ &= \tfrac{1}{3}\{(\sigma_I^2 - 2\sigma_I\sigma_{II} + \sigma_{II}^2) + (\sigma_{II}^2 - 2\sigma_{II}\sigma_{III} + \sigma_{III}^2) + (\sigma_{III}^2 - 2\sigma_{III}\sigma_I + \sigma_I^2)\}^{1/2} \\ &= \tfrac{1}{3}\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}\end{aligned}$$

2.23. The stress tensor at a point is given by $\sigma_{ij} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{pmatrix}$. Determine the maximum shear stress at the point and show that it acts in the plane which bisects the maximum and minimum stress planes.

From (*2.38*) the reader should verify that the principal stresses are $\sigma_I = 10$, $\sigma_{II} = 5$, $\sigma_{III} = -15$. From (*2.54b*) the maximum shear value is $\sigma_S = (\sigma_{III} - \sigma_I)/2 = -12.5$. The principal axes $Ox_1^*x_2^*x_3^*$ are related to the axes of maximum shear $Ox_1'x_2'x_3'$ by the transformation table below and are situated as shown in Fig. 2-26.

	x_1^*	x_2^*	x_3^*
x_1'	$1/\sqrt{2}$	0	$1/\sqrt{2}$
x_2'	0	1	0
x_3'	$-1/\sqrt{2}$	0	$1/\sqrt{2}$

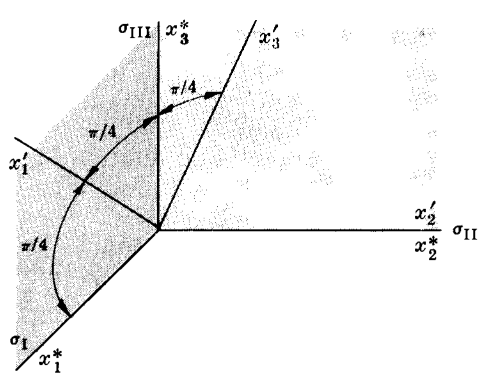


Fig. 2-26

The stress tensor referred to the primed axes is thus given by

$$[\sigma'_{ij}] = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -15 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2.5 & 0 & -12.5 \\ 0 & 5 & 0 \\ -12.5 & 0 & -2.5 \end{bmatrix}$$

The results here may be further clarified by showing the stresses on infinitesimal cubes at the point whose sides are perpendicular to the coordinate axes (see Fig. 2-27).

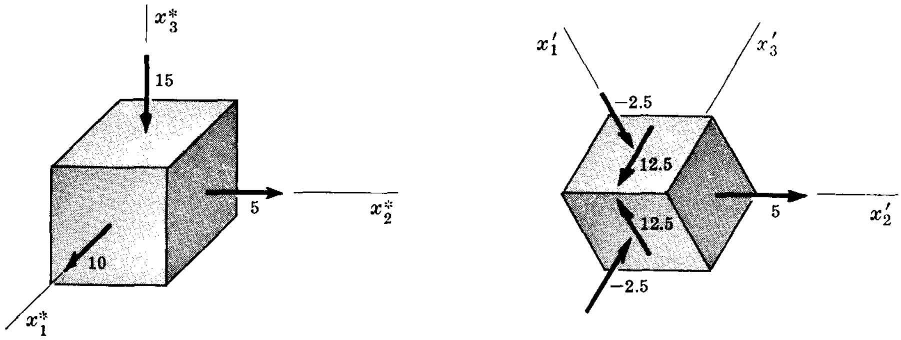


Fig. 2-27

MOHR'S CIRCLES (Sec. 2.12-2.13)

2.24. Draw the Mohr's circles for the state of stress discussed in Problem 2.23. Label important points. Relate the axes $Ox_1x_2x_3$ (conjugate to σ_{ij}) to the principal axes $Ox_1^*x_2^*x_3^*$ and locate on the Mohr's diagram the points giving the stress states on the coordinate planes of $Ox_1x_2x_3$.

The upper half of the symmetric Mohr's circles diagram is shown in Fig. 2-28 with the maximum shear point P and principal stresses labeled. The transformation table of direction cosines is

	x_1^*	x_2^*	x_3^*
x_1	0	1	0
x_2	−3/5	0	4/5
x_3	4/5	0	3/5

from which a diagram of the relative orientation of the axes is made as shown in Fig. 2-29. The x_1 and x_2^* axes are coincident. x_2 and x_3 are in the plane of $x_1^*x_3^*$ as shown. From the angles $\alpha = 36.8°$ and $\beta = 53.2°$ shown, the points $A(-6, 12)$ on the plane $\perp$ to x_2 and $B(1, 12)$ on the plane $\perp$ to x_3 are located. Point $C(5, 0)$ represents the stress state on the plane $\perp$ to x_1.

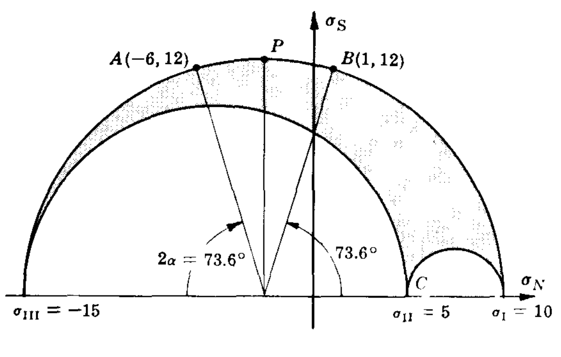


Fig. 2-28

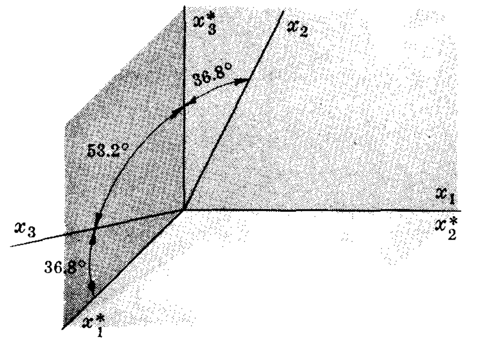


Fig. 2-29

2.25. The state of stress at a point referred to axes $Ox_1x_2x_3$ is given by

$$\sigma_{ij} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{pmatrix}$$

Determine analytically the stress vector components on the plane whose unit normal is $\hat{\mathbf{n}} = (2/3)\hat{\mathbf{e}}_1 + (1/3)\hat{\mathbf{e}}_2 + (2/3)\hat{\mathbf{e}}_3$. Check the results by the Mohr's diagram for this problem.

From (*2.13*) and the symmetry property of the stress tensor, the stress vector on the plane of $\hat{\mathbf{n}}$ is given by the matrix product

$$\begin{bmatrix} -5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -10/3 \\ -10 \\ -10/3 \end{bmatrix}$$

Thus $\mathbf{t}^{(\hat{\mathbf{n}})} = -10\,\hat{\mathbf{e}}_1/3 - 10\,\hat{\mathbf{e}}_2 - 10\,\hat{\mathbf{e}}_3/3$; and from (*2.33*), $\sigma_N = \mathbf{t}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = -70/9$. From (*2.47*), $\sigma_S = 70.7/9$.

For σ_{ij} the principal stress values are $\sigma_I = 10$, $\sigma_{II} = -5$, $\sigma_{III} = -15$; and the principal axes are related to $Ox_1x_2x_3$ by the table

	x_1	x_2	x_3
x_1^*	0	−3/5	4/5
x_2^*	1	0	0
x_3^*	0	4/5	3/5

Thus in the principal axes frame, $n_i^* = a_{ij}n_j$ or $\begin{bmatrix} 0 & -3/5 & 4/5 \\ 1 & 0 & 0 \\ 0 & 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. Accordingly the angles of Fig. 2-16 are given by $\theta = \beta = \cos^{-1} 2/3 = 48.2°$ and $\phi = \cos^{-1} 1/3 = 70.5°$, and the Mohr's diagram comparable to Fig. 2-17 is as shown in Fig. 2-30.

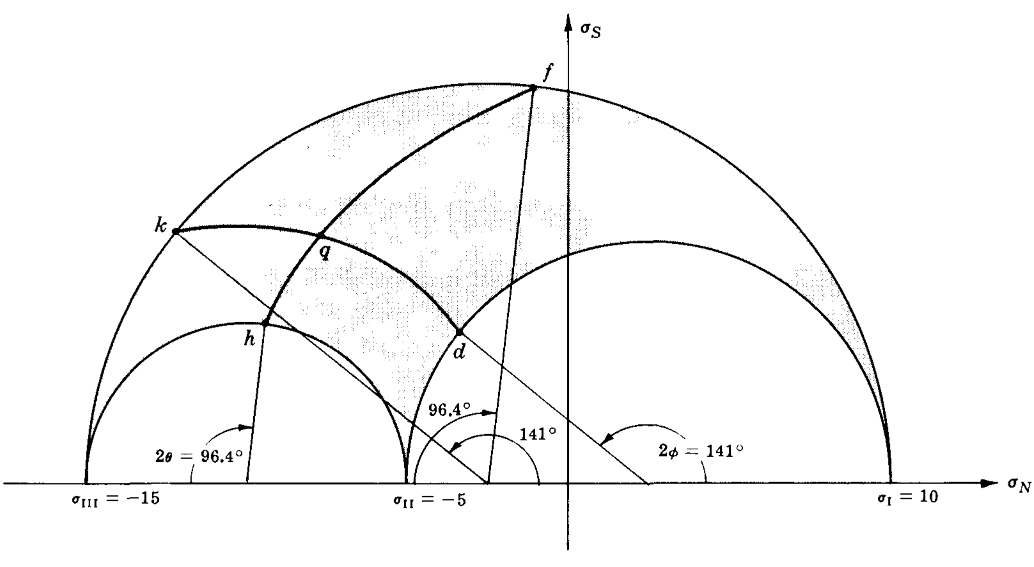


Fig. 2-30

2.26. Sketch the Mohr's circles for the three cases of plane stress depicted by the stresses on the small cube oriented along the coordinate axes shown in Fig. 2-31. Determine the maximum shear stress in each case.

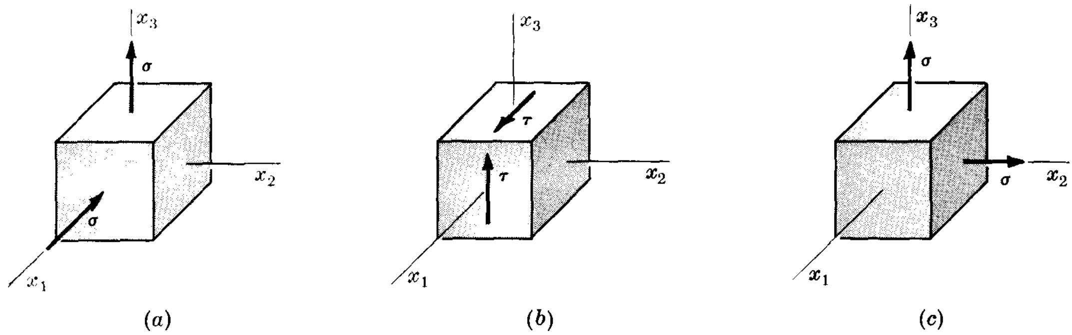


Fig. 2-31

The Mohr's circles are shown in Fig. 2-32.

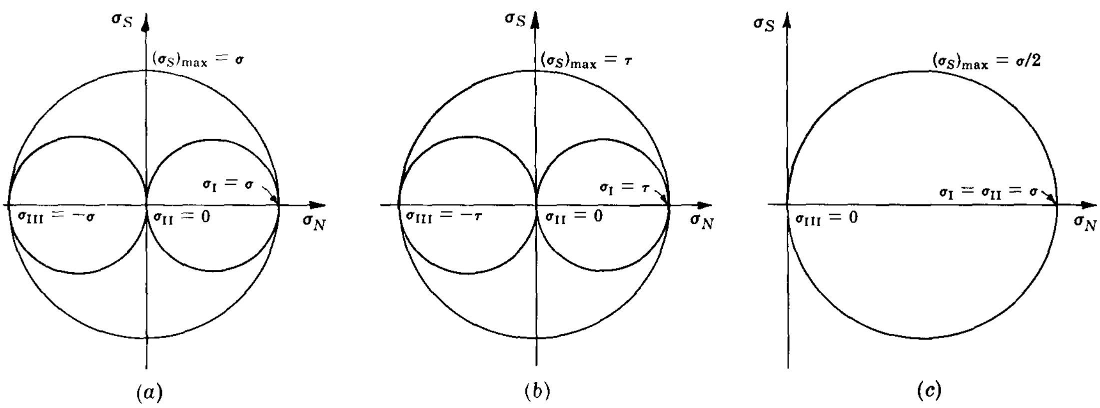


Fig. 2-32

SPHERICAL AND DEVIATOR STRESS (Sec. 2.14)

2.27. Split the stress tensor $\sigma_{ij} = \begin{pmatrix} 12 & 4 & 0 \\ 4 & 9 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ into its spherical and deviator parts and show that the first invariant of the deviator is zero.

$\sigma_M = \sigma_{kk}/3 = (12+9+3)/3 = 8$. Thus

$$\sigma_{ij} = \sigma_M \delta_{ij} + s_{ij} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 0 \\ 4 & 1 & -2 \\ 0 & -2 & -5 \end{pmatrix}$$

and $s_{ii} = 4 + 1 - 5 = 0$.

2.28. Show that the deviator stress tensor is equivalent to the superposition of five simple shear states.

The decomposition is

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} 0 & s_{12} & 0 \\ s_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & s_{13} \\ 0 & 0 & 0 \\ s_{31} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_{23} \\ 0 & s_{32} & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} s_{11} & 0 & 0 \\ 0 & -s_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -s_{33} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}$$

where the last two tensors are seen to be equivalent to simple shear states by comparison of cases (*a*) and (*b*) in Problem 2.26. Also note that since $s_{ii} = 0$, $-s_{11} - s_{33} = s_{22}$.

2.29. Determine the principal deviator stress values for the stress tensor

$$\sigma_{ij} = \begin{pmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The deviator of σ_{ij} is $s_{ij} = \begin{pmatrix} 3 & -6 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$ and its principal values may be determined from the determinant

$$\begin{vmatrix} 3-s & -6 & 0 \\ -6 & 3-s & 0 \\ 0 & 0 & -6-s \end{vmatrix} = (-6-s)(s+3)(s-9) = 0$$

Thus $s_{\rm I} = 9$, $s_{\rm II} = -3$, $s_{\rm III} = -6$. The same result is obtained by first calculating the principal stress values of σ_{ij} and then using *(2.71)*. For σ_{ij}, as the reader should show, $\sigma_{\rm I} = 16$, $\sigma_{\rm II} = 4$, $\sigma_{\rm III} = 1$ and hence $s_{\rm I} = 16 - 7 = 9$, $s_{\rm II} = 4 - 7 = -3$, $s_{\rm III} = 1 - 7 = -6$.

2.30. Show that the second invariant of the stress deviator is given in terms of its principal stress values by $\text{II}_{\Sigma_D} = (s_{\rm I}s_{\rm II} + s_{\rm II}s_{\rm III} + s_{\rm III}s_{\rm I})$, or by the alternative form $\text{II}_{\Sigma_D} = -\frac{1}{2}(s_{\rm I}^2 + s_{\rm II}^2 + s_{\rm III}^2)$.

In terms of the principal deviator stresses the characteristic equation of the deviator stress tensor is given by the determinant

$$\begin{vmatrix} s_{\rm I}-s & 0 & 0 \\ 0 & s_{\rm II}-s & 0 \\ 0 & 0 & s_{\rm III}-s \end{vmatrix} = (s_{\rm I}-s)(s_{\rm II}-s)(s_{\rm III}-s) = 0$$
$$= s^3 + (s_{\rm I}s_{\rm II} + s_{\rm II}s_{\rm III} + s_{\rm III}s_{\rm I})s - s_{\rm I}s_{\rm II}s_{\rm III}$$

Hence from (2.72), $\text{II}_{\Sigma_D} = (s_{\rm I}s_{\rm II} + s_{\rm II}s_{\rm III} + s_{\rm III}s_{\rm I})$. Since $s_{\rm I} + s_{\rm II} + s_{\rm III} = 0$,

$$\text{II}_{\Sigma_D} = \tfrac{1}{2}(2s_{\rm I}s_{\rm II} + 2s_{\rm II}s_{\rm III} + 2s_{\rm III}s_{\rm I} - (s_{\rm I} + s_{\rm II} + s_{\rm III})^2) = -\tfrac{1}{2}(s_{\rm I}^2 + s_{\rm II}^2 + s_{\rm III}^2)$$

MISCELLANEOUS PROBLEMS

2.31. Prove that for any symmetric tensor such as the stress tensor σ_{ij}, the transformed tensor σ'_{ij} in any other coordinate system is also symmetric.

From *(2.27)*, $\sigma'_{ij} = a_{ip}a_{jq}\sigma_{pq} = a_{jq}a_{ip}\sigma_{qp} = \sigma'_{ji}$.