Example 2.1 (dc Servo)

Description: A dc motor with constant field is driven by application of a voltage to its armature terminals. Through a set of gears, the motor drives a load with moment of inertia J, subject to an external torque (see Fig. 2.2). The control objective is to keep the load angle at some desired value.

Inputs and Outputs: The armature voltage v is the control input, and the load torque T is a disturbance. The outputs are the shaft angle θ and angular velocity ω.

Basic Principles: The gear ratio N is the ratio of angles and velocities of the two shafts; the torques have the same ratio. The load shaft is the high-torque, low-velocity shaft. The torque T_m produced by a dc motor is given by $T_m = k_1 \phi i$, where k_1 is a constant and ϕ is the field intensity. The armature circuit has resistance R and inductance L, and the motion generates a back emf $k_2 \phi \omega_m$. It can be shown that $k_1 = k_2$ in Système International (SI) units. The rotor of the dc motor has inertia J_m.

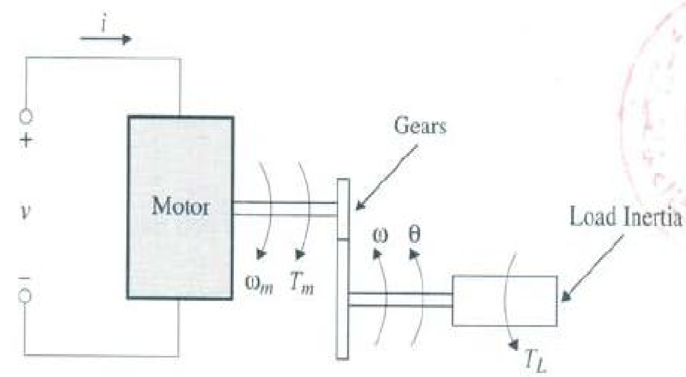

Figure 2.2 A dc servomechanism

Objectives: Write a model for the system, and simulate for the conditions given.

Solution Applying Newton's second law to the rotor,

$$J_m \dot{\omega}_m = T_m - T_e$$

where T_e is the torque exerted on the motor shaft by the load, transmitted through the gears. Thus,

$$T_e = T_m - J_m \dot{\omega}_m$$

so the torque at the motor shaft is seen to be the torque generated by the motor, minus the torque required to accelerate the rotor.

The torque exerted by the motor on the load shaft, transmitted through the gears, is NT_e. Newton's second law applied to the load is

$$\begin{aligned} J\dot{\omega} &= NT_e - T_L \\ &= NT_m - T_L - NJ_m\dot{\omega}_m. \end{aligned}$$

Since $\omega_m = N\omega$, this becomes

$$(J + N^2 J_m)\dot{\omega} = NT_m - T_L$$

or

$$J_e\dot{\omega} = NT_m - T_L \tag{2.13}$$

where $J_e = J + N^2 J_m$ is the effective inertia seen at the load shaft. With ϕ constant, let $k_1\phi = k_2\phi = K_m$. Then

$$T_m = K_m i$$

so that

$$\dot{\omega} = \frac{NK_m}{J_e} i - \frac{T_L}{J_e}. \tag{2.14}$$

To apply Equation 2.14, we need the current, i. By Kirchhoff's voltage law,

$$L\frac{di}{dt} + Ri = v - K_m \omega_m$$

where $K_m \omega$ is the back emf. This becomes

$$\dot{i} = -\frac{R}{L} i + \frac{v}{L} - \frac{NK_m}{L} \omega \tag{2.15}$$

where the fact that $\omega_m = N\omega$ has been used. Because the angle θ is of interest, the definition equation

$$\dot{\theta} = \omega \tag{2.16}$$

is added. Equations 2.14, 2.15, and 2.16 are the desired equations. In matrix form,

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \dfrac{NK_m}{J_e} \\ 0 & -\dfrac{NK_m}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{J_e} \\ \dfrac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v \\ T_L \end{bmatrix}. \tag{2.17}$$

Since θ and ω have been specified as the outputs, the output equation is

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$$\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix}. \tag{2.18}$$

For the specific values $K_m = .05$ Nm/A, $R = 1.2\ \Omega$, $L = .05$ H, $J_m = 8 \times 10^{-4}$ kg m^2, $J = 0.020$ kg m^2, and $N = 12$, the state equations are

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4.438 \\ 0 & -12 & -24 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -7.396 \\ 20 & 0 \end{bmatrix} \begin{bmatrix} v \\ T_L \end{bmatrix}. \tag{2.19}$$

The simulation conditions are as follows: with zero initial conditions and $T_L = 0$, $v(t) = 3$ V for $0 \le t \le 2$ and -3 V for $2 < t \le 4$. The result (MATLAB command lsim) is shown in Figure 2.3.

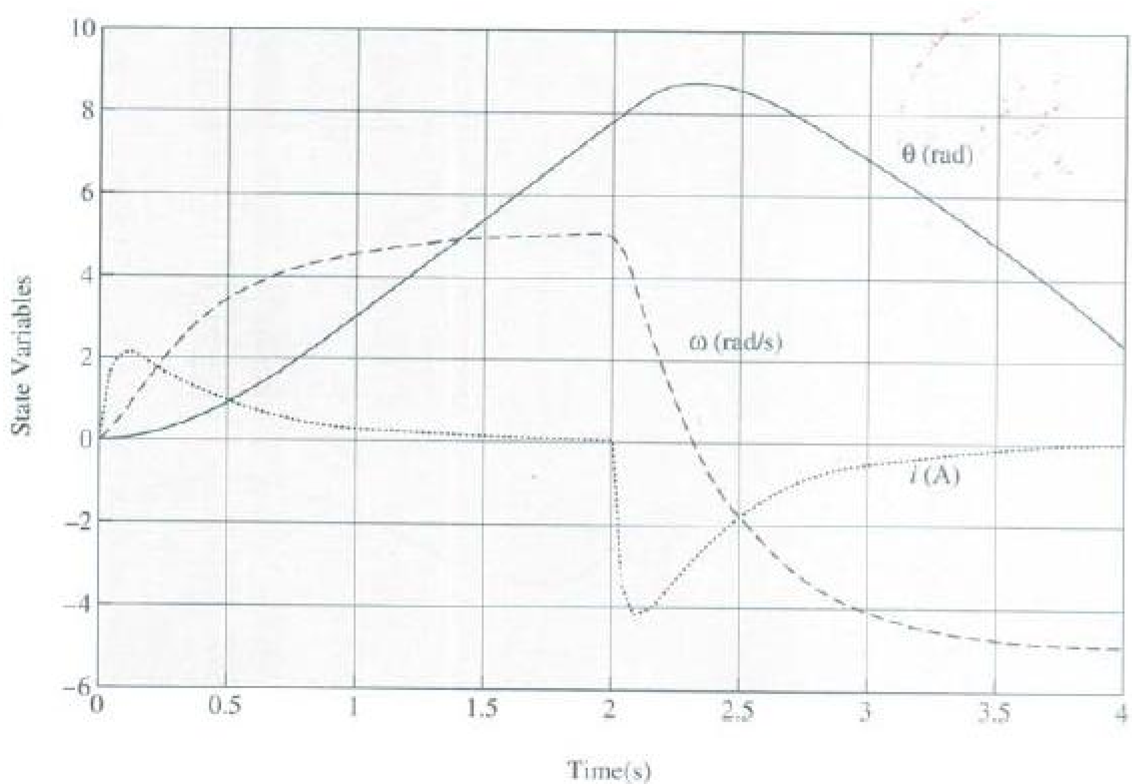


Figure 2.3 Time responses for the dc servo

1.

Problem

Servo with flexible shaft The low-velocity side of the gear box in Example 2.1 drives an inertial load through a shaft sufficiently long to exhibit torsional flexibility. The model of Figure 2.19 illustrates the situation: the spring is linear and develops a torque $K(\theta_1 - \theta_2)$.

a. Model this system. Suggested steps:

i. With $\omega_2 = \dot{\theta}_2$, $J\dot{\omega}_2 =$ torque from spring.

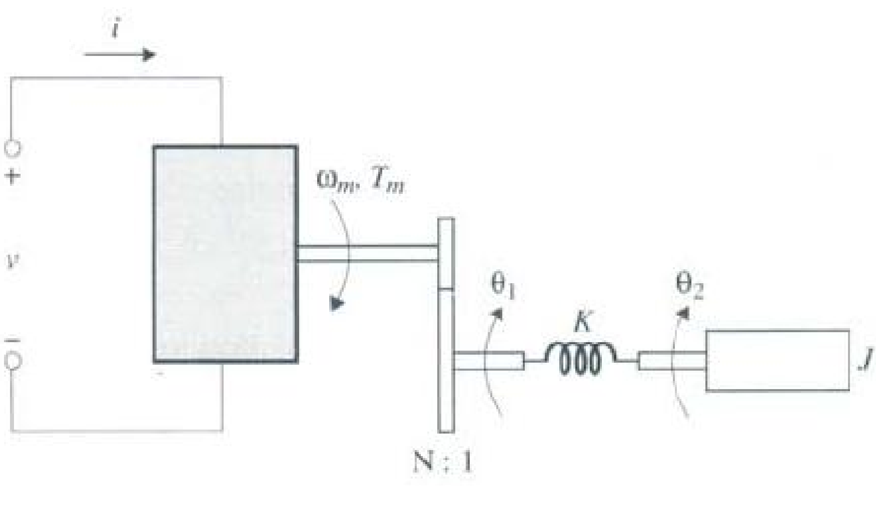
ii. $J_m\dot{\omega}_m = T_m - \frac{1}{N}$ (torque exerted by spring).

iii. Write di/dt as in Example 2.1, and use the fact that $\omega_m = N\dot{\theta}_1$.

iv. Because $\theta_1 - \theta_2$ is small, the equations for $\dot{\omega}_1$ and $\dot{\omega}_2$ call for differences of almost equal terms. It is numerically preferable to work with $\Delta = \theta_1 - \theta_2$. Define $\dot{\Delta} = \Omega$, and write an equation for $\dot{\Omega}$ in terms of Δ and i. Write the state equations, using the state variables $\theta_2, \Delta, \omega_2, \Omega$, and i, with v as the input.

b. Write the state equations for the specific values of Example 2.1, with $K = 500$ Nm/rad.

c. Simulate under the conditions given in Example 2.1.



Servo with flexible shaft The dc servo of Problem 2.5 (Chapter 2) (or, equivalently, of Problem 3.14 in Chapter 3) has two pairs of complex, underdamped poles. One way of controlling such poles is by placing LHP complex zeros relatively near; since closed-loop poles migrate to zeros as the gain is increased, zeros "attract" the underdamped poles away from the j-axis.

Consider the compensator

$$F(s) = k\frac{(s + 100 \pm j100)(s + 5 \pm j5)}{(s + 200)^4}.$$

a. Obtain the Root Locus.

b. Calculate the range of k for stability.

c. Calculate (roughly) the value of k for which the lowest of the damping factors associated with any pair of complex poles is maximized; i.e., maximize the smallest damping factor.

d. For k as in part (c), compute the closed-loop step response.

Servo with flexible shaft The transfer function θ_2/v of the servo with flexible shaft was computed in Problem 3.14 (Chapter 3). From the Bode plot, calculate the range of values, if any, of the gain k of a pure-gain controller for which the closed-loop system is stable.