Structural Dynamics

Lecture Eleven: Dynamic Response of MDOF Systems: (Chapter 11)

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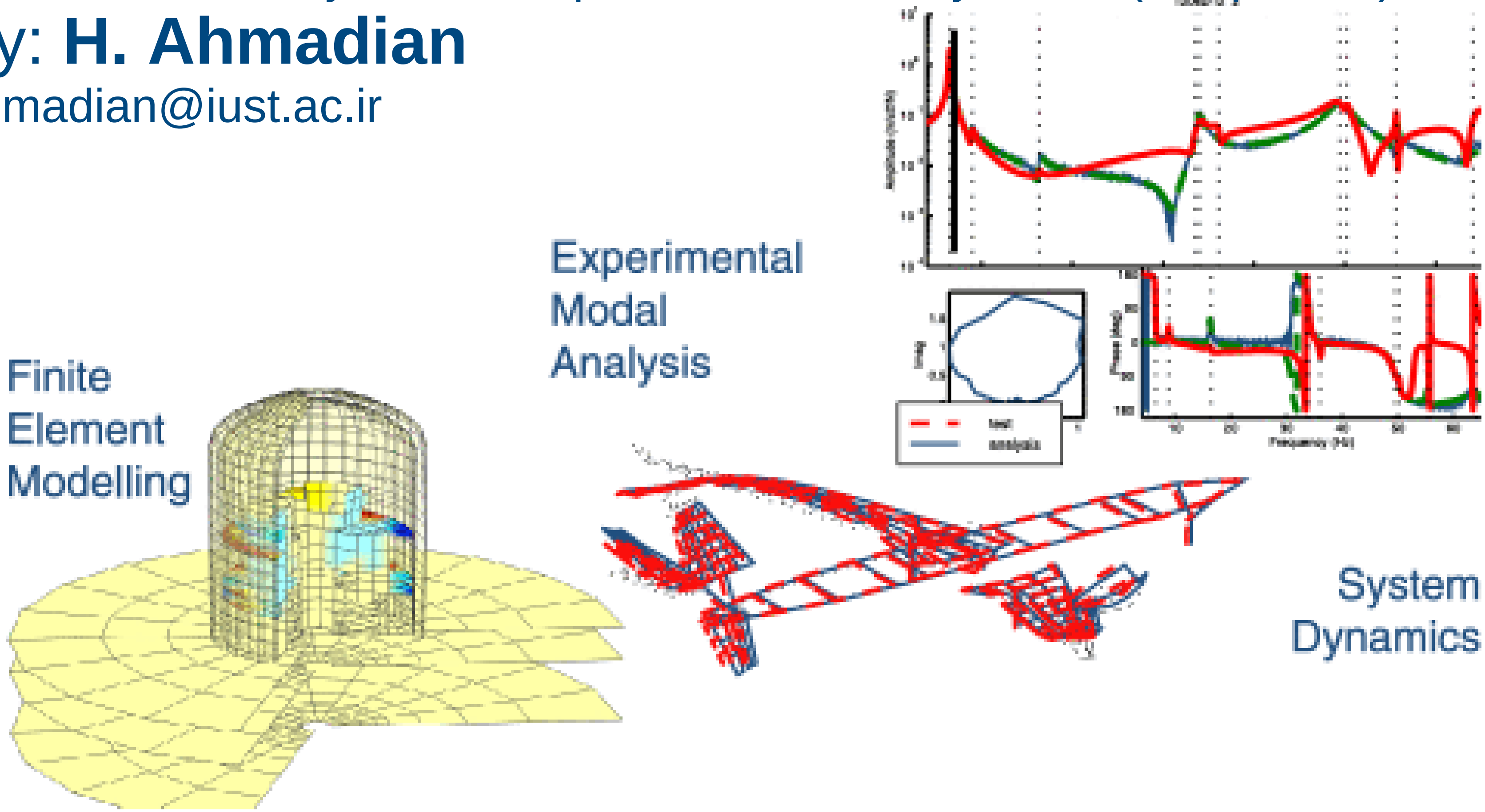

School of Mechanical Engineering
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Dynamic Response of MDOF Systems: Mode-Superposition Method

- Mode-Superposition Method: Principal Coordinates
- Mode-Superposition Solutions for MDOF Systems with Modal Damping: Frequency-Response Analysis
- Mode-Displacement Solution for the Response of MDOF Systems
- Mode-Acceleration Solution for the Response of Undamped MDOF Systems
- Dynamic Stresses by Mode Superposition
- Mode Superposition for Undamped Systems with Rigid-Body Modes

MODE-DISPLACEMENT SOLUTION FOR THE RESPONSE OF MDOF SYSTEMS

$$\boxed{\mathbf{u}(t) = \mathbf{\Phi}\boldsymbol{\eta}(t) = \sum_{r=1}^{N} \boldsymbol{\phi}_r \eta_r(t)}$$

$$\boxed{\begin{aligned}\eta_r(t) &= \frac{1}{M_r\omega_{dr}} \int_0^t f_r(\tau)\, e^{-\zeta_r\omega_r(t-\tau)} \sin\omega_{dr}(t-\tau)\, d\tau \\ &\quad + e^{-\zeta_r\omega_r t}\left[\eta_r(0)\cos\omega_{dr}t + \frac{\dot{\eta}_r(0) + \zeta_r\omega_n\eta_r(0)}{\omega_{dr}} \sin\omega_d t\right]\end{aligned}}$$

- In many cases only a subset of the modes is available
- We examine modal truncation and determine the factors to be considered in deciding how many modes to include.

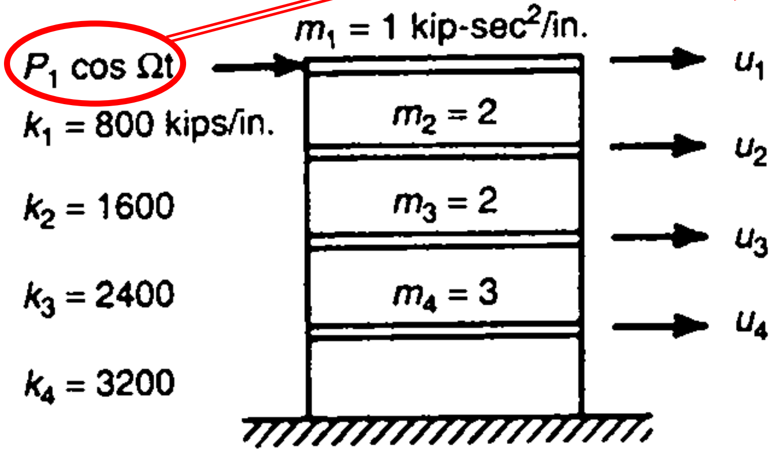
Mode-displacement solution

$$\boxed{\hat{\mathbf{u}}(t) = \hat{\boldsymbol{\Phi}}\,\hat{\boldsymbol{\eta}}(t) = \sum_{r=1}^{\hat{N}} \boldsymbol{\phi}_r \eta_r(t)}$$

$$\hat{\boldsymbol{\Phi}} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_{\hat{N}}] \qquad \hat{N} < N$$

- The mode-displacement solution ignores completely the contribution of modes not included in the set
- The "kept“ modes are not restricted to the lowest-frequency modes if some modes of higher frequency are available and are considered to be important.

Example

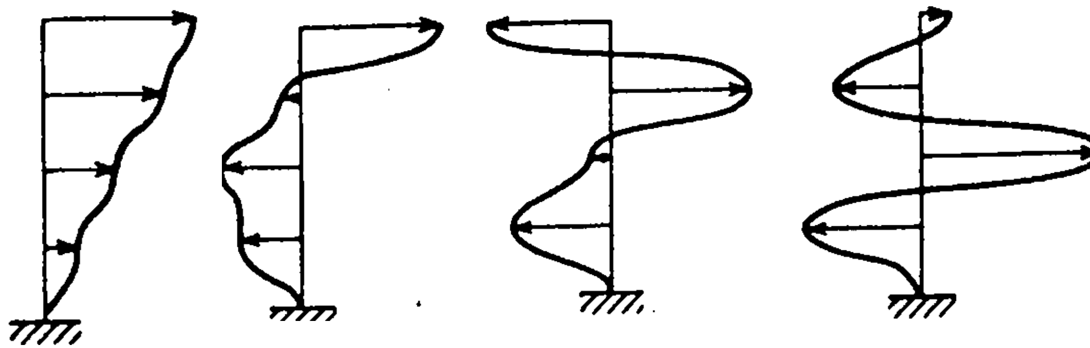


excitation frequencies $\Omega = 0$, $\Omega = 0.5\omega_1$, and $\Omega = 1.3\omega_3$

$$\mathbf{K} = 800 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 7 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\omega^2 = \begin{Bmatrix} 0.17672 \\ 0.87970 \\ 1.68746 \\ 3.12279 \end{Bmatrix} \times 10^3,$$

$$\omega = \begin{Bmatrix} 13.294 \\ 29.660 \\ 41.079 \\ 55.882 \end{Bmatrix} \text{rad/sec}$$



What conclusions can you draw concerning truncation to one mode, to two modes, or to three modes?

Example: Modal masses and stiffnesses

$$\Phi = \begin{bmatrix} 1.00000 & 1.00000 & -0.90145 & 0.15436 \\ 0.77910 & -0.09963 & 1.00000 & -0.44817 \\ 0.49655 & -0.53989 & -0.15859 & 1.00000 \\ 0.23506 & -0.43761 & -0.70797 & -0.63688 \end{bmatrix}$$

$M_1 = 2.87290$ kip-sec^2/in., $K_1 = 507.691$ kips/in.

$M_2 = 2.17732$ kip-sec^2/in., $K_2 = 1915.39$ kips/in.

$M_3 = 4.36660$ kip-sec^2/in., $K_3 = 7368.45$ kips/in.

$M_4 = 3.64239$ kip-sec^2/in., $K_4 = 11,374.4$ kips/in.

Example: The modal forces

$$\mathbf{p}(t) = \mathbf{P}\cos\Omega t = \begin{Bmatrix} P_1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \cos\Omega t$$

$$F_r = \boldsymbol{\phi}_r^T \mathbf{P}$$

$$F_1 = P_1$$

$$F_2 = P_1$$

$$F_3 = -0.90145 P_1$$

$$F_4 = 0.15436 P_1$$

Example: The mode-displacement

$$\eta_r(t) = \frac{F_r}{K_r}\frac{1}{1-(\Omega/\omega_r)^2}\cos\Omega t \qquad \hat{u}_1(t) = \sum_{r=1}^{\hat{N}} \phi_{1r}\eta_r(t)$$

$$\left.\left.\left.\begin{aligned} u_1(t) = {} & \left.\frac{1.0(P_1\cos\Omega t)}{507.691[1-(\Omega^2/176.72)]}\right] \hat{N}=1 \\ & + \frac{1.0(P_1\cos\Omega t)}{1915.39[1-(\Omega^2/879.70)]} \end{aligned}\right] \hat{N}=2 \\ & + \frac{-0.90145(-0.90145\,P_1\cos\Omega t)}{7368.45[1-(\Omega^2/1687.46)]} \right] \hat{N}=3 \\ & + \frac{0.15436(0.15436\,P_1\cos\Omega t)}{11,374.4[1-(\Omega^2/3122.79)]}$$

Example: The mode-displacement

Constant C in $u_1(t) = C P_1 \cos \Omega t$:

	$\widehat{N} = 1$	$\widehat{N} = 2$	$\widehat{N} = 3$	$\widehat{N} = 4$
$\Omega = 0$	$1.970(10^{-3})$	$2.492(10^{-3})$	$2.602(10^{-3})$	$2.604(10^{-3})$
$\Omega = 0.5\,\omega_1$	$2.626(10^{-3})$	$3.176(10^{-3})$	$3.289(10^{-3})$	$3.291(10^{-3})$
$\Omega = 1.3\,\omega_3$	$-1.301(10^{-3})$	$-3.630(10^{-3})$	$-5.228(10^{-3})$	$-4.987(10^{-3})$

1. A one-mode solution is not accurate at any of the three frequencies.
2. A three-mode solution is quite accurate for $\Omega = 0$ and for $\Omega = 0.5\omega_1$, but since $\Omega = 1.3\omega_3$ is almost equal to ω_4, an important contribution to $u_1(t)$ at this frequency is the mode 4 contribution. A truncated solution is useless at this frequency.

Mode-acceleration solution

$$\mathbf{u} = \mathbf{K}^{-1}[\mathbf{p}(t) - \mathbf{C}\dot{\mathbf{u}} - \mathbf{M}\ddot{\mathbf{u}}]$$

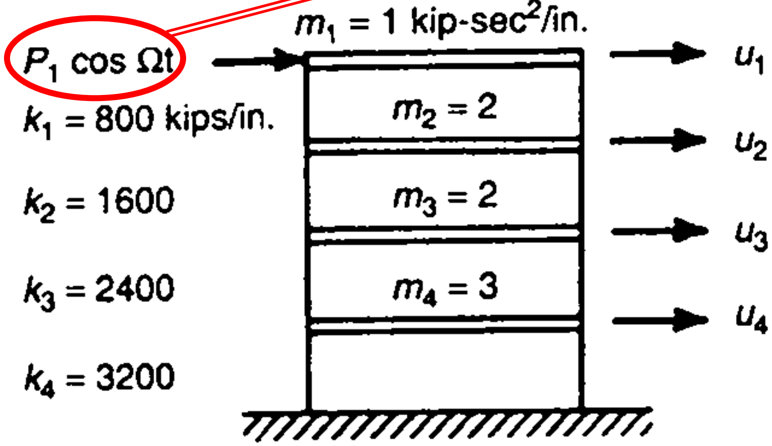
$$\mathbf{u} = \mathbf{K}^{-1}[\mathbf{p}(t) - \mathbf{C}\boldsymbol{\Phi}\dot{\eta} - \mathbf{M}\boldsymbol{\Phi}\ddot{\eta}]$$

$$\boxed{\tilde{\mathbf{u}}(t) = \mathbf{K}^{-1}\mathbf{p}(t) - \sum_{r=1}^{\hat{N}} \frac{2\zeta_r}{\omega_r}\phi_r\dot{\eta}_r(t) - \sum_{r=1}^{\hat{N}} \frac{1}{\omega_r^2}\phi_r\ddot{\eta}_r(t)}$$

$$\hat{N} < N$$

Example

- Determine an expression for u_1 using Mode Acceleration Method,
- Compare the results with the results of mode-displacement method.



excitation frequencies $\Omega = 0$, $\Omega = 0.5\omega_1$, and $\Omega = 1.3\omega_3$

$$\mathbf{A} \equiv \mathbf{K}^{-1} = \begin{bmatrix} 2.60417 & 1.35417 & 0.72917 & 0.31250 \\ 1.35417 & 1.35417 & 0.72917 & 0.31250 \\ 0.72917 & 0.72917 & 0.72917 & 0.31250 \\ 0.31250 & 0.31250 & 0.31250 & 0.31250 \end{bmatrix} (10^3) \text{ in./kip}$$

Example: The mode-acceleration solution

$$\tilde{\mathbf{u}}(t) = \mathbf{K}^{-1}\mathbf{p}(t) - \sum_{r=1}^{\hat{N}} \frac{1}{\omega_r^2} \boldsymbol{\phi}_r \ddot{\eta}_r(t)$$

$$\mathbf{p}(t) = \mathbf{P}\cos\Omega t \quad F_r(t) = \boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{P}$$

$$\eta_r(t) = \frac{F_r}{K_r} \frac{1}{1-(\Omega/\omega_r)^2} \cos\Omega t$$

$$\tilde{u}_1 = a_{11} P_1 \cos\Omega t - \sum_{r=1}^{\hat{N}} \frac{1}{\omega_r^2} \phi_{1r} \ddot{\eta}_r$$

$$\tilde{u}_1 = a_{11} P_1 \cos\Omega t + \sum_{r=1}^{\hat{N}} \frac{\Omega_r^2}{\omega_r^2} \phi_{1r} \frac{F_r}{K_r} \frac{1}{1-(\Omega/\omega_r)^2} \cos\Omega t$$

Example: The mode-acceleration solution

$$
\left.\left.\left.\begin{aligned}
u_1(t) = {} & 2.60417(10^{-3})(P_1\cos\Omega t) \\
& + \frac{(\Omega^2/176.72)(1.0)(P_1\cos\Omega t)}{507.695[1-(\Omega^2/176.72)]}
\end{aligned}\right]\hat{N}=1 \\
+ \frac{(\Omega^2/879.70)(1.0)(P_1\cos\Omega t)}{1915.39[1-(\Omega^2/879.70)]}
\right]\hat{N}=2 \\
+ \frac{(\Omega^2/1687.46)(-0.90145)(-0.90145\,P_1\cos\Omega t)}{7368.43[1-(\Omega^2/1687.46)]}
\right]\hat{N}=3 \\
+ \frac{(\Omega^2/3122.79)(0.15436)(0.15436\,P_1\cos\Omega t)}{11,374.4[1-(\Omega^2/3122.79)]}
$$

Example: The mode-acceleration solution

Constant C in $u_1(t) = CP_1 \cos \Omega t$:

	$\widehat{N} = 1$	$\widehat{N} = 2$	$\widehat{N} = 3$	$\widehat{N} = 4$
$\Omega = 0$	$2.604(10^{-3})$	$2.604(10^{-3})$	$2.604(10^{-3})$	$2.604(10^{-3})$
$\Omega = 0.5\,\omega_1$	$3.261(10^{-3})$	$3.288(10^{-3})$	$3.291(10^{-3})$	$3.291(10^{-3})$
$\Omega = 1.3\,\omega_1$	$5.044(10^{-3})$	$-2.506(10^{-3})$	$-5.207(10^{-3})$	$-4.987(10^{-3})$

(c) From the foregoing table we can conclude that:

1. The exact static solution is produced at $\Omega = 0$ without any contribution from normal modes.
2. At the low frequency of $\Omega = 0.5\,\omega_1$, even a one-term solution is fairly accurate, and the mode-acceleration solution is accurate to two places for $\widehat{N} = 2$, as compared with $\widehat{N} = 3$ for the mode-displacement solution.
3. Since the forcing frequency $\Omega = 1.3\,\omega_3$ lies between ω_3 and ω_4, a truncated mode-acceleration solution is not any better than a truncated mode-displacement solution—the fourth mode is needed in either case.

DYNAMIC STRESSES BY MODE SUPERPOSITION

For the mode-displacement method, the internal stresses are given by:

$$\boxed{\hat{\boldsymbol{\sigma}}(t) = \sum_{r=1}^{\hat{N}} \mathbf{s}_r \eta_r(t)}$$

For the mode-acceleration method for an undamped system, the displacement approximation leads to the stress approximation:

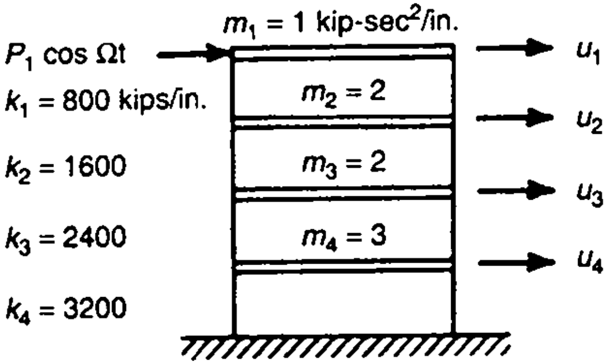
$$\boxed{\tilde{\boldsymbol{\sigma}}(t) \doteq \boldsymbol{\sigma}_{\text{pseudostatic}} - \sum_{r=1}^{\hat{N}} \frac{1}{\omega_r^2} \mathbf{s}_r \ddot{\eta}_r(t)}$$

Example

- For the shear building, write an expression for the shear force at the i^th story corresponding to mode *r*.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & 0 & k_3 & -k_3 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \qquad \begin{Bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{Bmatrix}_r = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & 0 & k_3 & -k_3 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}_r$$

$$\mathbf{s}_1 = \begin{Bmatrix} 176.72 \\ 452.08 \\ 627.58 \\ 752.19 \end{Bmatrix} \quad \mathbf{s}_2 = \begin{Bmatrix} 879.70 \\ 704.42 \\ -245.47 \\ -1400.35 \end{Bmatrix}$$



$$\mathbf{s}_3 = \begin{Bmatrix} -1521.16 \\ 1853.74 \\ 1318.51 \\ -2265.50 \end{Bmatrix} \quad \mathbf{s}_4 = \begin{Bmatrix} 482.02 \\ -2317.07 \\ 3928.51 \\ -2038.02 \end{Bmatrix}$$

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Lecture 12: Dynamic Response of MDOF Systems: (Chapter 11)

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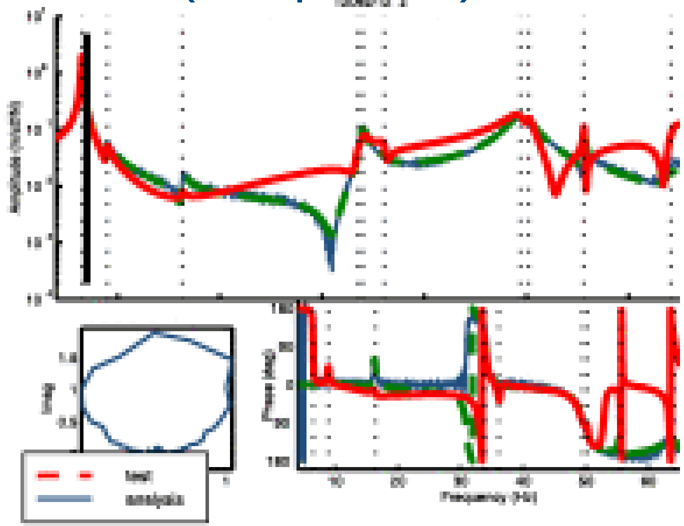
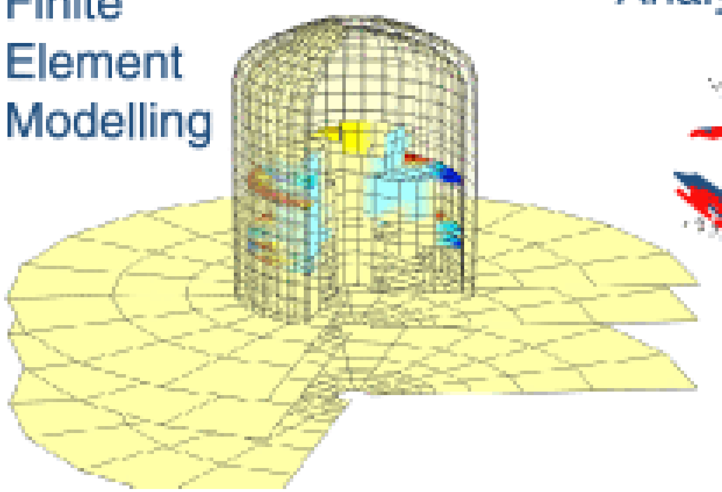
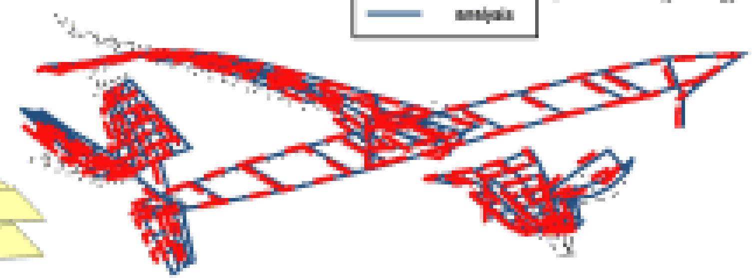


Experimental
Modal
Analysis

Finite
Element
Modelling





System
Dynamics

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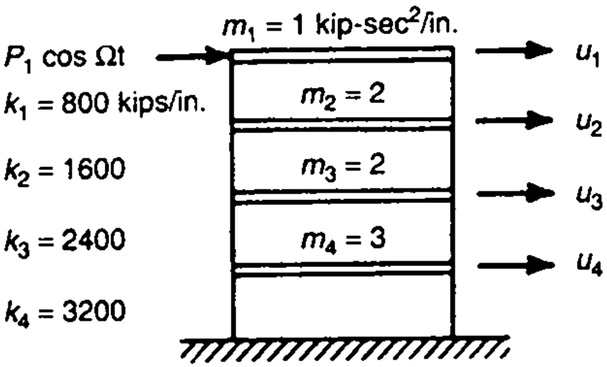
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & 0 & k_3 & -k_3 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\begin{Bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 \\ 0 & 0 & k_3 & -k_3 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix},$$

$$\mathbf{s}_1 = \begin{Bmatrix} 176.72 \\ 452.08 \\ 627.58 \\ 752.19 \end{Bmatrix} \quad \mathbf{s}_2 = \begin{Bmatrix} 879.70 \\ 704.42 \\ -245.47 \\ -1400.35 \end{Bmatrix}$$



$$\mathbf{s}_3 = \begin{Bmatrix} -1521.16 \\ 1853.74 \\ 1318.51 \\ -2265.50 \end{Bmatrix} \quad \mathbf{s}_4 = \begin{Bmatrix} 482.02 \\ -2317.07 \\ 3928.51 \\ -2038.02 \end{Bmatrix}$$

MODE SUPERPOSITION FOR UNDAMPED SYSTEMS WITH RIGID-BODY MODES

$$\mathbf{u}(t) = \mathbf{u}_R(t) + \mathbf{u}_E(t) = \mathbf{\Phi}_R \eta_R(t) + \mathbf{\Phi}_E \eta_E(t)$$

$$M_R \ddot{\eta}_R = \mathbf{\Phi}_R^T \mathbf{p}(t)$$

$$\boxed{\eta_r(t) = \int_0^t \int_0^\tau \frac{1}{M_r} f_r(\xi)\, d\xi\, d\tau + t\, \dot{\eta}_r(0) + \eta_r(0), \qquad r = 1, 2, \ldots, N_R}$$

$$M_E \ddot{\eta}_E + K_E \eta_E = \mathbf{\Phi}_E^T \mathbf{p}(t)$$

$$\boxed{\begin{aligned} \eta_r(t) &= \eta_r(0) \cos \omega_r t + \frac{1}{\omega_r} \dot{\eta}_r(0) \sin \omega_r t \\ &\quad + \frac{1}{M_r \omega_r} \int_0^t \boldsymbol{\phi}_r^T \mathbf{p}(\tau) \sin \omega_r (t - \tau)\, d\tau \end{aligned}}$$

Mode-Displacement Method for Systems with Rigid-Body Modes

- All rigid-body modes are employed,
- Included are number of elastic modes.

$$\boxed{\hat{\mathbf{u}}(t) = \mathbf{\Phi}_R \eta_R(t) + \hat{\mathbf{\Phi}}_E \hat{\eta}_E(t)}$$

- Rigid-body displacements do not give rise to internal stresses:

$$\hat{\boldsymbol{\sigma}}(t) = \hat{\mathbf{S}}_E \hat{\eta}_E = \sum_{r=1}^{\hat{N}_E} \mathbf{s}_r \eta_r(t)$$

Mode-Acceleration Method for Systems with Rigid-Body Modes

- The stiffness matrix is singular and cannot be inverted,
- The mode-acceleration method cannot be employed in the straightforward manner

$$\mathbf{u}(t) = \mathbf{\Phi}_R \boldsymbol{\eta}_R(t) + (\mathbf{\Phi}_E K_E^{-1} \mathbf{\Phi}_E^T)\mathbf{p}(t) - (\mathbf{\Phi}_E K_E^{-1} M_E)\ddot{\boldsymbol{\eta}}_E$$

- Truncating the number of elastic modes

$$\boxed{\tilde{\mathbf{u}}(t) = \mathbf{\Phi}_R \boldsymbol{\eta}_R(t) + \mathbf{A}_E \mathbf{p}(t) - (\hat{\mathbf{\Phi}}_E \hat{K}_E^{-1} \hat{M}_E)\ddot{\hat{\boldsymbol{\eta}}}_E}$$

$$\mathbf{A}_E = \mathbf{\Phi}_E K_E^{-1} \mathbf{\Phi}_E^T$$

Must include all flexible modes.

Mode-Acceleration Method for Systems with Rigid-Body Modes

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

$$\mathbf{u}(t) = \mathbf{u}_R(t) + \mathbf{u}_E(t)$$

$$\mathbf{M}\ddot{\mathbf{u}}_E + \mathbf{K}\mathbf{u}_E = \mathbf{p}_E(t)$$

$$\mathbf{p}_E(t) = \mathbf{p}(t) - \mathbf{M}\ddot{\mathbf{u}}_R$$

In determining the elastic displacements, we use a self-equilibrated force system of applied forces and rigid-body inertia forces.

Mode-Acceleration Method for Systems with Rigid-Body Modes

$$\mathbf{p}_E(t) = \mathbf{p}(t) - \mathbf{M}\ddot{\mathbf{u}}_R$$

$$\ddot{\mathbf{u}}_R = \mathbf{\Phi}_R \ddot{\eta}_R = \mathbf{\Phi}_R M_R^{-1} \mathbf{\Phi}_R^T \mathbf{p}(t)$$

$$\mathbf{p}_E(t) = \mathbf{R}\mathbf{p}(t)$$

$$\boxed{\mathbf{R} = \mathbf{I} - \mathbf{M}\mathbf{\Phi}_R M_R^{-1} \mathbf{\Phi}_R^T}$$

The *Inertia-Relief Matrix*

Mode-Acceleration Method for Systems with Rigid-Body Modes

- In order to calculate a flexibility matrix we need to impose N_R arbitrary constraints.
- Then let $\boldsymbol{A}_R$ be the flexibility matrix of the system relative to these statically determinate constraints,
  - with zeros filling in the N_R rows and columns corresponding to the constraints.

$$\mathbf{w} = \mathbf{A}_R \mathbf{p}_E$$

Mode-Acceleration Method for Systems with Rigid-Body Modes

$$\mathbf{w}_E = \mathbf{w} - \boldsymbol{\Phi}_R \mathbf{c}_R$$

$$\boldsymbol{\Phi}_R^T \mathbf{M} \mathbf{w}_E = \mathbf{0}. \longrightarrow \mathbf{c}_R = M_R^{-1} \boldsymbol{\Phi}_R^T \mathbf{M} \mathbf{w}$$

$$\mathbf{w}_E = (\mathbf{I} - \boldsymbol{\Phi}_R M_R^{-1} \boldsymbol{\Phi}_R^T \mathbf{M})\mathbf{w} = \mathbf{R}^T \mathbf{w}$$

$$\mathbf{w}_E = \mathbf{A}_E \mathbf{p}$$

$$\mathbf{A}_E = \mathbf{R}^T \mathbf{A}_R \mathbf{R}$$

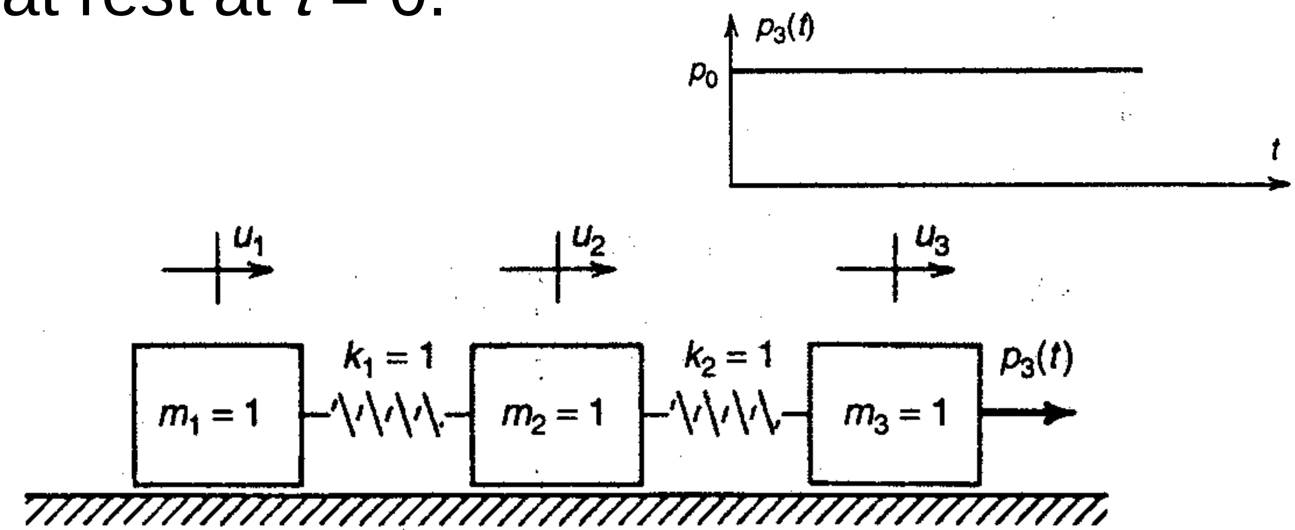
$$\boxed{\tilde{\mathbf{u}}(t) = \boldsymbol{\Phi}_R \eta_R(t) + \mathbf{R}^T \mathbf{A}_R \mathbf{R} \mathbf{p}(t) - (\hat{\boldsymbol{\Phi}}_E \hat{K}_E^{-1} \hat{M}_E) \ddot{\hat{\boldsymbol{\eta}}}_E(t)}$$

Stresses in Truncated Models of Systems with Rigid-Body Modes

$$\tilde{\sigma}(t) = \sigma_{\text{pseudostatic}} - \sum_{r=1}^{\hat{N}_E} \frac{1}{\omega_r^2} \mathbf{s}_r \ddot{\eta}_r(t)$$

Stresses in Truncated Models of Systems with Rigid-Body Modes

Example 11.8 Use the mode-displacement method and the mode-acceleration method to determine expressions for the maximum force in each of the two springs shown in Fig.1 due to application of a step force *P3(t)* = *Po. t >0*. Compare the convergence of the two methods. The system is at rest at *t* = 0.



Example 11.8 Modes and Natural Frequencies

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{u} = \boldsymbol{\phi} \cos \omega t$$

$$\begin{bmatrix} 1-\omega^2 & -1 & 0 \\ -1 & 2-\omega^2 & -1 \\ 0 & -1 & 1-\omega^2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\omega^2(\omega^4 - 4\omega^2 + 3) = 0$$

Example 11.8 Modes and Natural Frequencies

$$\omega_1^2 = 0, \qquad \omega_2^2 = 1, \qquad \omega_3^2 = 3$$

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \qquad \boldsymbol{\phi}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \qquad \boldsymbol{\phi}_3 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

$$M_r = \boldsymbol{\phi}_r^{\mathrm{T}} \mathbf{M} \boldsymbol{\phi}_r = \boldsymbol{\phi}_r^{\mathrm{T}} \boldsymbol{\phi}_r, \qquad K_r = \omega_r^2 M_r$$

$$M_1 = 3, \qquad M_2 = 2, \qquad M_3 = 6$$

$$K_1 = 0, \qquad K_2 = 2, \qquad K_3 = 18$$

Example 11.8 Initial conditions and generalized forces

$$\eta_r(0) = \dot{\eta}_r(0) = 0, \qquad r = 1, 2, 3$$

$$f_r(t) = \boldsymbol{\phi}_r^{\mathrm{T}} \mathbf{p}(t)$$

$$f_1(t) = p_3(t) = p_0,$$

$$f_2(t) = -p_3(t) = -p_0,$$

$$f_3(t) = p_3(t) = p_0$$

Example 11.8 Solutions in modal coordinates

$$\left.\begin{aligned} 3\ddot{\eta}_1 &= p_0 \\ 2\ddot{\eta}_2 + 2\eta_2 &= -p_0 \\ 6\ddot{\eta}_3 + 18\eta_3 &= p_0 \end{aligned}\right\} \qquad \begin{aligned} \eta_1 &= \frac{p_0 t^2}{6} \\ \eta_2 &= \frac{-p_0}{2}(1 - \cos\omega_2 t) \\ \eta_3 &= \frac{p_0}{18}(1 - \cos\omega_3 t) \end{aligned}$$

Example 11.8 The *modal stress vectors*

$$\mathbf{s}_r = \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix}_r = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}_r$$

$$\mathbf{s}_2 = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix}, \qquad \mathbf{s}_3 = \begin{Bmatrix} -3 \\ 3 \end{Bmatrix}$$

Example 11.8 The mode-displacement approximation to the spring forces (internal stresses)

$$\hat{\boldsymbol{\sigma}}_{(\text{one-mode})} = \mathbf{s}_2 \eta_2(t)$$

$$\hat{\boldsymbol{\sigma}}_{(\text{one-mode})} = \begin{Bmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \frac{-p_0}{2}(1 - \cos \omega_2 t)$$

$$= \frac{p_0}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} (1 - \cos \omega_2 t)$$

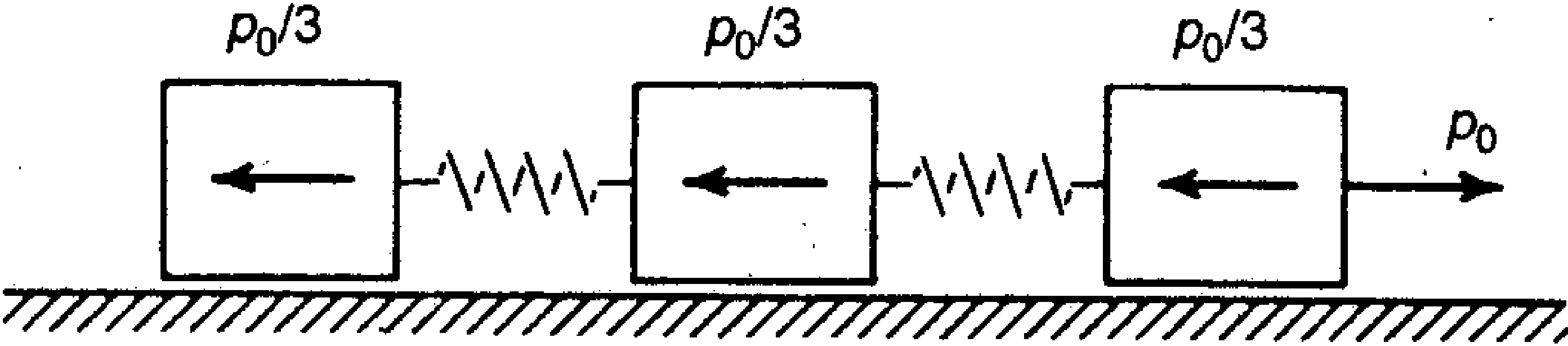
Example 11.8 The mode-displacement approximation to the spring forces (internal stresses)

$$\hat{\boldsymbol{\sigma}}_{(\text{two-mode})} = \boldsymbol{\sigma}(t) = \mathbf{s}_2\eta_2(t) + \mathbf{s}_3\eta_3(t)$$

$$\hat{\boldsymbol{\sigma}}_{(\text{two-mode})} = \boldsymbol{\sigma} \equiv \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} = \frac{p_0}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}(1 - \cos\omega_2 t)$$

$$+ \frac{p_0}{6}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}(1 - \cos\omega_3 t)$$

Example 11.8 The mode-acceleration solution for internal stresses



Self-equilibrating force system due to rigid-body motion.

$$\sigma_{\text{pseudostatic}} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix}_{\text{pseudostatic}} = \begin{Bmatrix} \dfrac{p_0}{3} \\ \dfrac{2p_0}{3} \end{Bmatrix}$$

Example 11.8 The mode-acceleration solution for internal stresses

$$\tilde{\boldsymbol{\sigma}}(t) = \boldsymbol{\sigma}_{\text{pseudostatic}} - \frac{1}{\omega_2^2}\mathbf{s}_2\ddot{\eta}_2(t)$$

$$\tilde{\boldsymbol{\sigma}}_{(\text{one-mode})} = \begin{Bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \end{Bmatrix} = \frac{p_0}{3}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} - \frac{p_0}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\cos\omega_2 t$$

$$\tilde{\boldsymbol{\sigma}}(t) = \boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{pseudostatic}} - \frac{1}{\omega_2^2}\mathbf{s}_2\ddot{\eta}_2(t) - \frac{1}{\omega_3^2}\mathbf{s}_3\ddot{\eta}_3(t)$$

$$\tilde{\boldsymbol{\sigma}}_{(\text{two-mode})} = \boldsymbol{\sigma} \equiv \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} = \frac{p_0}{3}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$- \frac{p_0}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\cos\omega_2 t + \frac{p_0}{6}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\cos\omega_3 t$$

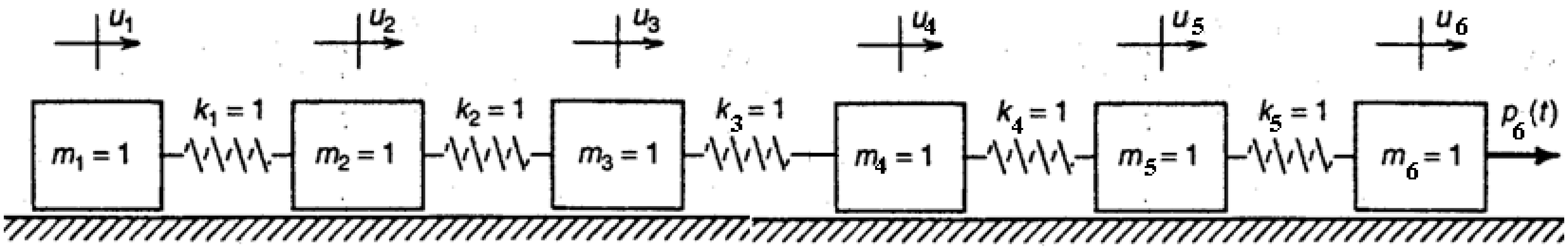
Example 11.8 Comparison

	Mode-displacement one-mode	*Mode-acceleration one-mode*	*Exact*[a]
σ_1/p_0	1.000000	0.833333	0.999933
σ_1/p_0	1.000000	1.166667	1.333241

[a] "Exact" values computed by evaluating σ_1 and σ_2 from Eq. 18*b* at 1° intervals to $\omega t = 100\pi$.

- The two mode solution are identical, since this system has only two elastic modes.
- The example is too small to indicate improved "convergence" of the mode acceleration method over the mode-displacement method.

Extra Example



$$K := \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2-\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2+\sqrt{3} \end{bmatrix}$$

$$\Phi := \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{1}{3\sqrt{2}-\sqrt{6}} & \frac{1}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{6} & -\frac{1}{3\sqrt{2}+\sqrt{6}} \\ \frac{\sqrt{6}}{6} & \frac{1-\sqrt{3}}{3\sqrt{2}-\sqrt{6}} & 0 & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3\sqrt{2}+\sqrt{6}} \\ \frac{\sqrt{6}}{6} & \frac{-2+\sqrt{3}}{3\sqrt{2}-\sqrt{6}} & \frac{-1}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{6} & \frac{-2-\sqrt{3}}{3\sqrt{2}+\sqrt{6}} \\ \frac{\sqrt{6}}{6} & \frac{2-\sqrt{3}}{3\sqrt{2}-\sqrt{6}} & \frac{-1}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{6} & \frac{2+\sqrt{3}}{3\sqrt{2}+\sqrt{6}} \\ \frac{\sqrt{6}}{6} & \frac{-1+\sqrt{3}}{3\sqrt{2}-\sqrt{6}} & 0 & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} & \frac{-1-\sqrt{3}}{3\sqrt{2}+\sqrt{6}} \\ \frac{\sqrt{6}}{6} & \frac{1}{3\sqrt{2}-\sqrt{6}} & \frac{1}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{6} & \frac{1}{3\sqrt{2}+\sqrt{6}} \end{bmatrix}$$

Extra Example

```
> Ar:=copyinto(inverse(submatrix(K,2..6,2..6)),Matrix(6,6),2,2);
```

$$Ar := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 4 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

```
> R:=map(simplify,matadd(Matrix(6,6,shape=identity),-multiply(multiply(M,submatrix(Phi,1..6,1
transpose(submatrix(Phi,1..6,1..1)))));
```

$$R := \begin{bmatrix} \frac{5}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} \\ \frac{-1}{6} & \frac{5}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} \\ \frac{-1}{6} & \frac{-1}{6} & \frac{5}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} \\ \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{5}{6} & \frac{-1}{6} & \frac{-1}{6} \\ \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{5}{6} & \frac{-1}{6} \\ \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{-1}{6} & \frac{5}{6} \end{bmatrix}$$

Extra Example

```
> Ae:=multiply(transpose(R),multiply(Ar,R));
```

$$Ae := \begin{bmatrix} \frac{55}{36} & \frac{25}{36} & \frac{1}{36} & \frac{-17}{36} & \frac{-29}{36} & \frac{-35}{36} \\ \frac{25}{36} & \frac{31}{36} & \frac{7}{36} & \frac{-11}{36} & \frac{-23}{36} & \frac{-29}{36} \\ \frac{1}{36} & \frac{7}{36} & \frac{19}{36} & \frac{1}{36} & \frac{-11}{36} & \frac{-17}{36} \\ \frac{-17}{36} & \frac{-11}{36} & \frac{1}{36} & \frac{19}{36} & \frac{7}{36} & \frac{1}{36} \\ \frac{-29}{36} & \frac{-23}{36} & \frac{-11}{36} & \frac{7}{36} & \frac{31}{36} & \frac{25}{36} \\ \frac{-35}{36} & \frac{-29}{36} & \frac{-17}{36} & \frac{1}{36} & \frac{25}{36} & \frac{55}{36} \end{bmatrix}$$

Dynamic Response of MDOF Systems: Mode-Superposition Method

- Mode-Superposition Method: Principal Coordinates
- Mode-Superposition Solutions for MDOF Systems with Modal Damping: Frequency-Response Analysis
- Mode-Displacement Solution for the Response of MDOF Systems
- Mode-Acceleration Solution for the Response of Undamped MDOF Systems
- Dynamic Stresses by Mode Superposition
- Mode Superposition for Undamped Systems with Rigid-Body Modes

Structural Dynamics

Lecture 13: Mathematical Models of Continuous Systems (Chapter 12)

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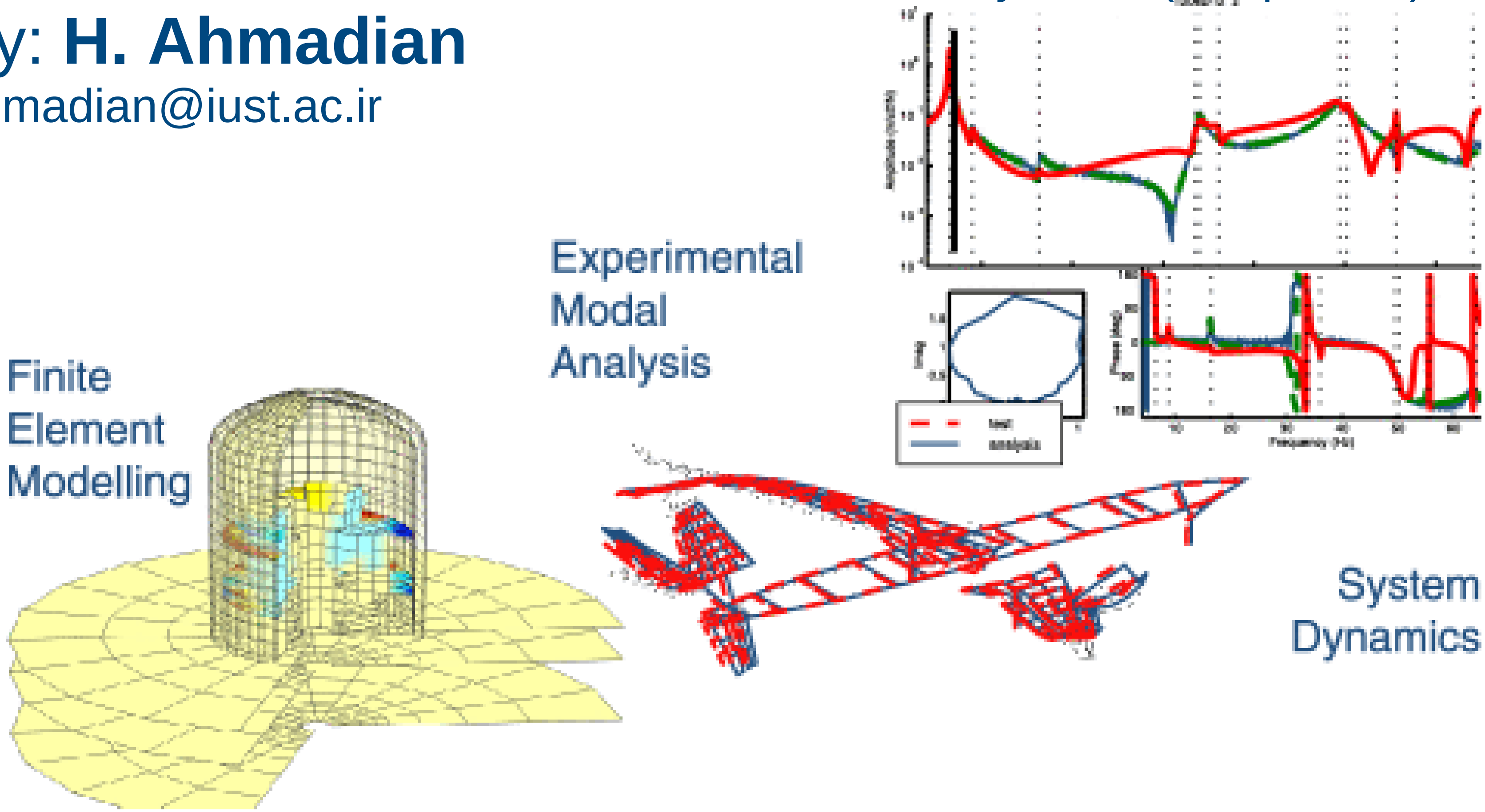


School of Mechanical Engineering

Iran University of Science and Technology

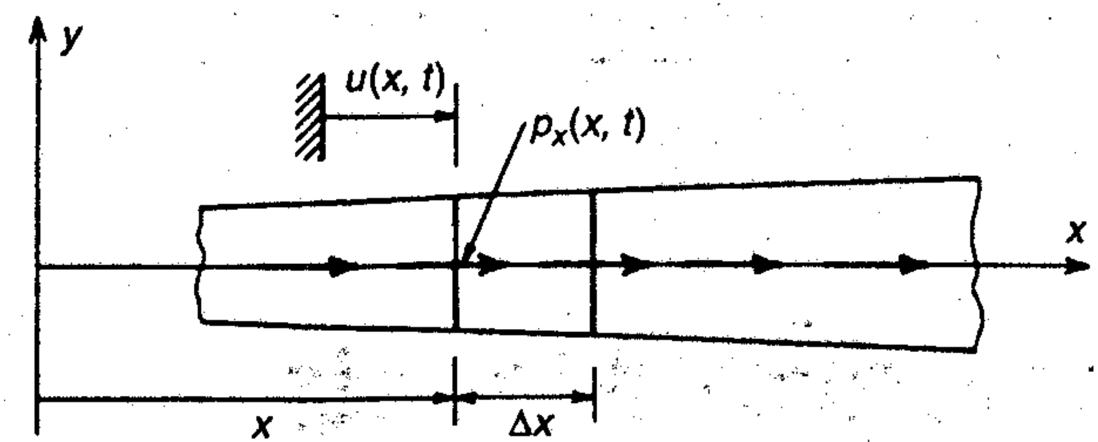
Mathematical Models of Continuous Systems

- Applications of Newton's Laws: Axial Deformation and Torsion
- Application of Newton's Laws: Transverse Vibration of Linearly Elastic Beams (Bernoulli-Euler Beam Theory)
- Application of Hamilton's Principle: Torsion of a Rod with Circular Cross Section
- Application of the Extended Hamilton's Principle: Beam Flexure Including Shear Deformation and Rotatory Inertia (Timoshenko Beam Theory)

APPLICATIONS OF NEWTON'S LAWS: AXIAL DEFORMATION AND TORSION

The *axial deformation assumptions,*

- The axis of the member remains straight.
- Cross sections remain plane and remain perpendicular to the axis of the member.
- The material is linearly elastic. '
- The material properties *(E, p)* are constant at a given cross section, but may vary with *x.*



Constitutive Equation

$$\sigma = E\epsilon$$

$$\epsilon(x,t) = \frac{\partial u(x,t)}{\partial x}$$

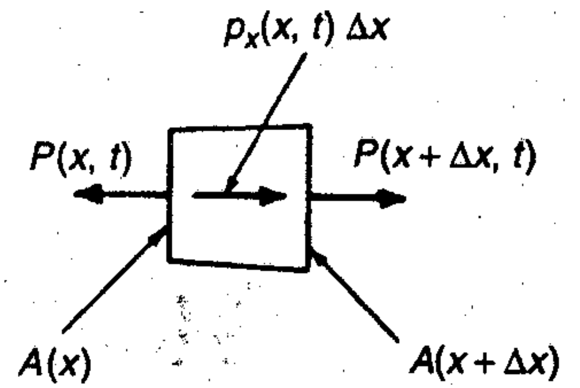
$$P(x,t) = \iint_A \sigma \, dA = A\sigma$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{P}{AE}}$$

Applications of Newton's Laws:

$$\overset{+}{\rightarrow} \sum F_x = \Delta m a_x$$

$$p_x \, \Delta x + P(x + \Delta x, t) - P(x, t) = \rho A \, \Delta x \, \frac{\partial^2 u}{\partial t^2}$$



$$\lim_{\Delta x \to 0} \frac{P(x + \Delta x, t) - P(x, t)}{\Delta x} + p_x(x, t) = \rho A \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial P}{\partial x} + p_x = \rho A \frac{\partial^2 u}{\partial t^2}$$

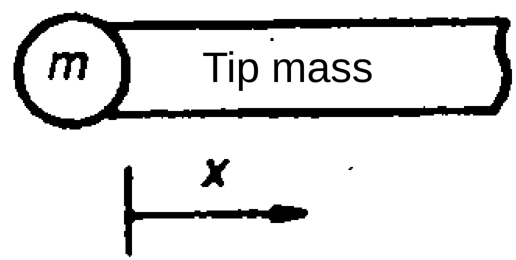
$$\boxed{\frac{\partial}{\partial x}\left(AE \frac{\partial u}{\partial x}\right) + p_x(x, t) = \rho A \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < L}$$

Common Boundary Conditions:

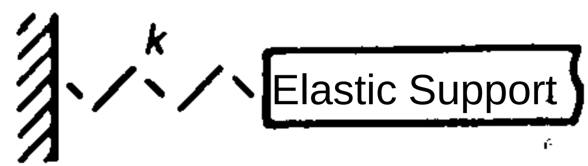
$$\boxed{u(x_e, t) = 0, \qquad \text{fixed end}}$$

$$\boxed{\left.\frac{\partial u}{\partial x}\right|_{x_e} = 0, \qquad \text{force-free end}}$$

$$\left.\left(AE\frac{\partial u}{\partial x}\right)\right|_{x=0} = m\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=0}$$

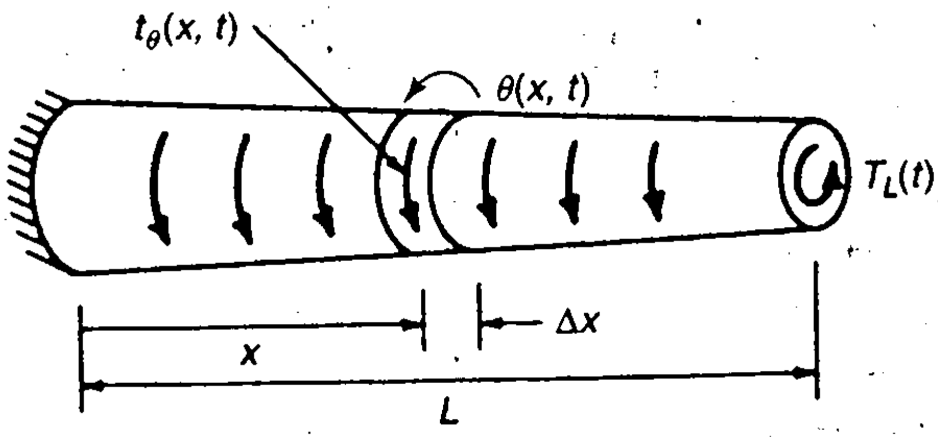


$$\left.\left(AE\frac{\partial u}{\partial x}\right)\right|_{x=0} = ku(0, t)$$



Torsional Deformation of Rods with Circular Cross Section

- The axis of the member, which is labeled the *x* axis, remains straight.
- Cross sections remain plane and remain perpendicular to the axis of the member.
- Radial lines in each cross section remain straight and radial as the cross section rotates through angle *e* about the axis.
- The material is linearly elastic $\tau = G\gamma$
- The shear modulus is constant at a given cross section but may vary with *x.*



Newton's Law for moments

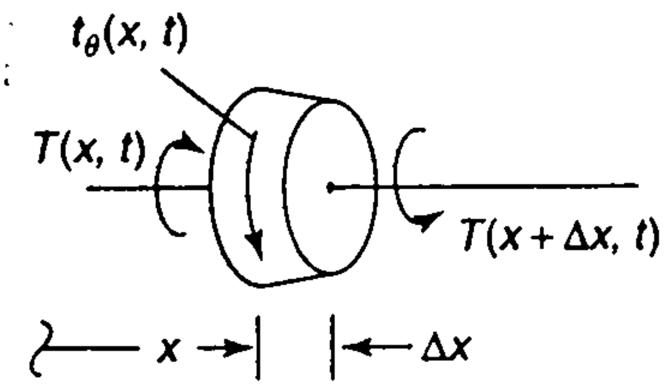
$$\sum M_x = (\rho I_p \, \Delta x) \frac{\partial^2 \theta}{\partial t^2}$$

$$t_\theta(x,t)\,\Delta x + T(x+\Delta x, t) - T(x,t) = (\rho I_p \, \Delta x) \frac{\partial^2 \theta}{\partial t^2}$$

$$\lim_{\Delta x \to 0} \frac{T(x+\Delta x, t) - T(x,t)}{\Delta x} + t_\theta(x,t) = \rho I_p \frac{\partial^2 \theta}{\partial t^2}$$

$$\frac{\partial T}{\partial x} + t_\theta = \rho I_p \frac{\partial^2 \theta}{\partial t^2}$$

$$\boxed{\frac{\partial \theta}{\partial x} = \frac{T}{G I_p}}$$



$$\boxed{\frac{\partial}{\partial x}\left(G I_p \frac{\partial \theta}{\partial x}\right) + t_\theta(x,t) = \rho I_p \frac{\partial^2 \theta}{\partial t^2}, \qquad 0 < x < L}$$

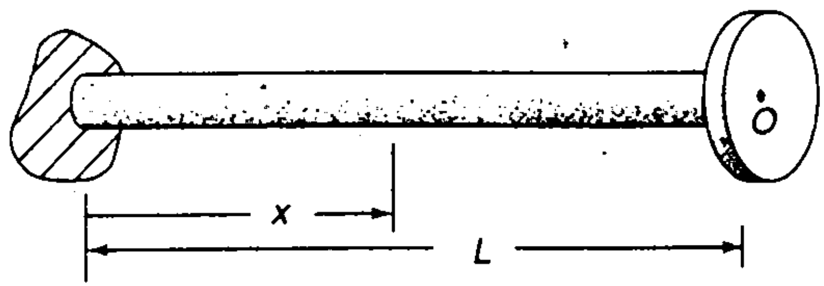
Common Boundary Conditions:

$$\boxed{\theta(x_e, t) = 0, \qquad \text{fixed end}}$$

$$\boxed{\left(GI_p \frac{\partial \theta}{\partial x}\right)\bigg|_{x_e} = T_e(t), \qquad \text{torque-loaded end}}$$

An example:

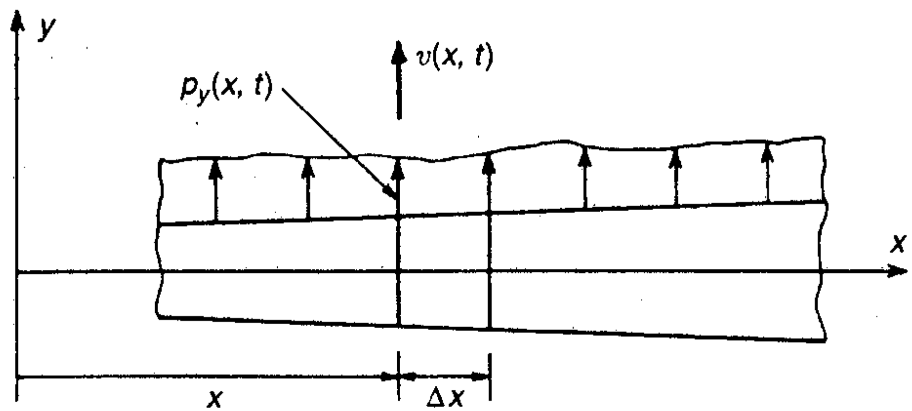
Rod-disk system $\left(GI_p \frac{\partial \theta}{\partial x}\right)\bigg|_{x=L} = -I_O \ddot{\theta}(L, t)$



TRANSVERSE VIBRATION OF LINEARLY ELASTIC EULER-BERNOULLI BEAMS

The *Euler-Bernoulli assumptions of elementary beam theory* are:

- The *x-y* plane is a principal plane of the beam, and it remains plane as the beam deforms in the *y* direction.
- There is an axis of the beam, which undergoes no extension or contraction. This is called the *neutral axis,* and it is labeled the *x* axis. The original *xz* plane is called the *neutral surface.'*
- Cross sections, which are perpendicular to the neutral axis in the undeformed beam, remain plane and remain perpendicular to the deformed neutral axis; that is, transverse shear deformation is neglected.
- The material is linearly elastic, with modulus of elasticity *E(x);* that is, the beam is homogeneous at any cross section. (Generally, *E* is constant throughout the beam.) .
- Normal stresses *along y* and *z* are negligible compared to that of *x*



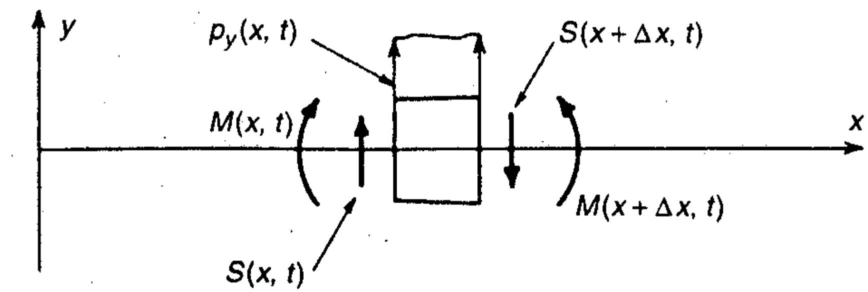
TRANSVERSE VIBRATION OF LINEARLY ELASTIC EULER-BERNOULLI BEAMS

The following dynamics assumptions will also be made:

- The rotatory inertia of the beam may be neglected in the moment equation.
- The mass density is constant at each cross section, so that the mass center coincides with the centroid of the cross section.

TRANSVERSE VIBRATION OF LINEARLY ELASTIC EULER-BERNOULLI BEAMS

$$M(x,t) = EI\,\frac{\partial^2 v}{\partial x^2}$$



$$\uparrow + \sum F_y = \Delta m a_y$$

$$S(x,t) - S(x+\Delta x,t) + p_y(x,t)\,\Delta x = \rho A\,\Delta x\,\frac{\partial^2 v}{\partial t^2}$$

$$-\frac{\partial S}{\partial x} + p_y(x,t) = \rho A\,\frac{\partial^2 v}{\partial t^2}$$

$$\circlearrowleft + \sum M_G = \cancel{\Delta I_G \alpha} \Longrightarrow S = \frac{\partial M}{\partial x}$$

$$\boxed{\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 v}{\partial x^2}\right) + \rho A\,\frac{\partial^2 v}{\partial t^2} = p_y(x,t), \qquad 0 < x < L}$$

Common Boundary Conditions:

Fixed end at $x = x_e$:

$$\boxed{v(x_e, t) = 0, \quad \text{and} \quad \left.\frac{\partial v}{\partial x}\right|_{x=x_e} = 0}$$

Simply supported end at $x = x_e$:

$$\boxed{v(x_e, t) = 0 \quad \text{and} \quad \left.\frac{\partial^2 v}{\partial x^2}\right|_{x=x_e} = 0}$$

Common Boundary Conditions:

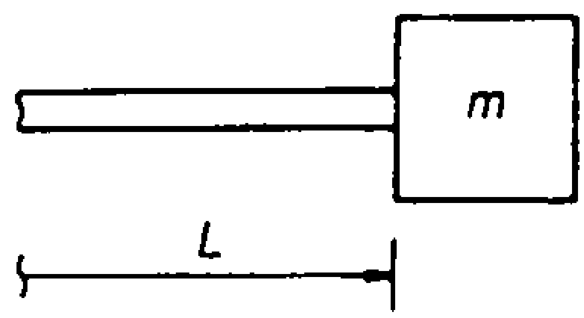
Free end

$$\boxed{\frac{\partial}{\partial x}\left(EI\,\frac{\partial^2 v}{\partial x^2}\right)\bigg|_{x=x_e} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}\bigg|_{x=x_e} = 0}$$

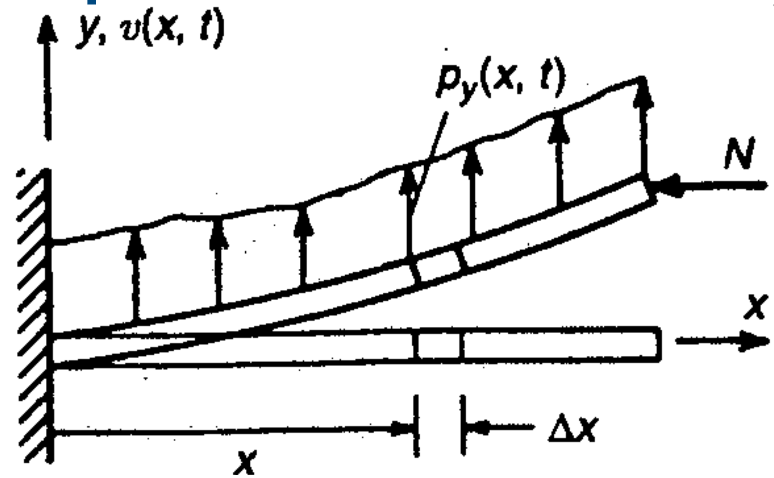
The tip mass at *x*=*L*:

$$\frac{\partial}{\partial x}\left(EI\,\frac{\partial^2 v}{\partial x^2}\right)\bigg|_{x=L} = m\,\frac{\partial^2 v}{\partial t^2}\bigg|_{x=L}$$

$$\frac{\partial^2 v}{\partial x^2}\bigg|_{x=L} = 0$$



A Beam Subjected to Compressive End Load



$$\uparrow + \sum F_y = m a_y$$

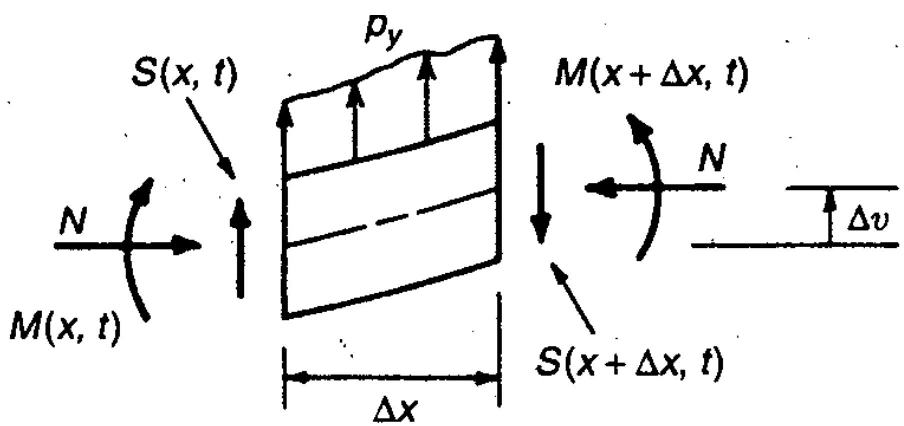
$$-\frac{\partial S}{\partial x} + p_y(x,t) = \rho A \frac{\partial^2 v}{\partial t^2}$$

$$\circlearrowleft + \sum M_G = 0$$

$$M(x+\Delta x, t) - M(x,t) + N[v(x+\Delta x, t) - v(x,t)] - S(x+\Delta x, t)\,\Delta x = 0$$

$$\frac{\partial M}{\partial x} + N\frac{\partial v}{\partial x} = S$$

$$\frac{\partial^2 M}{\partial x^2} + N\frac{\partial^2 v}{\partial x^2} + \rho A \frac{\partial^2 v}{\partial t^2} = p_y(x,t)$$



$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) + N\frac{\partial^2 v}{\partial x^2} + \rho A \frac{\partial^2 v}{\partial t^2} = p_y(x,t)$$

Mathematical Models of Continuous Systems

- Applications of Newton's Laws: Axial Deformation and Torsion
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- Application of the Extended Hamilton's Principle: Beam Flexure Including Shear Deformation and Rotatory Inertia (Timoshenko Beam Theory)

Structural Dynamics

Lecture 14: Mathematical Models of Continuous Systems (Chapter 12)

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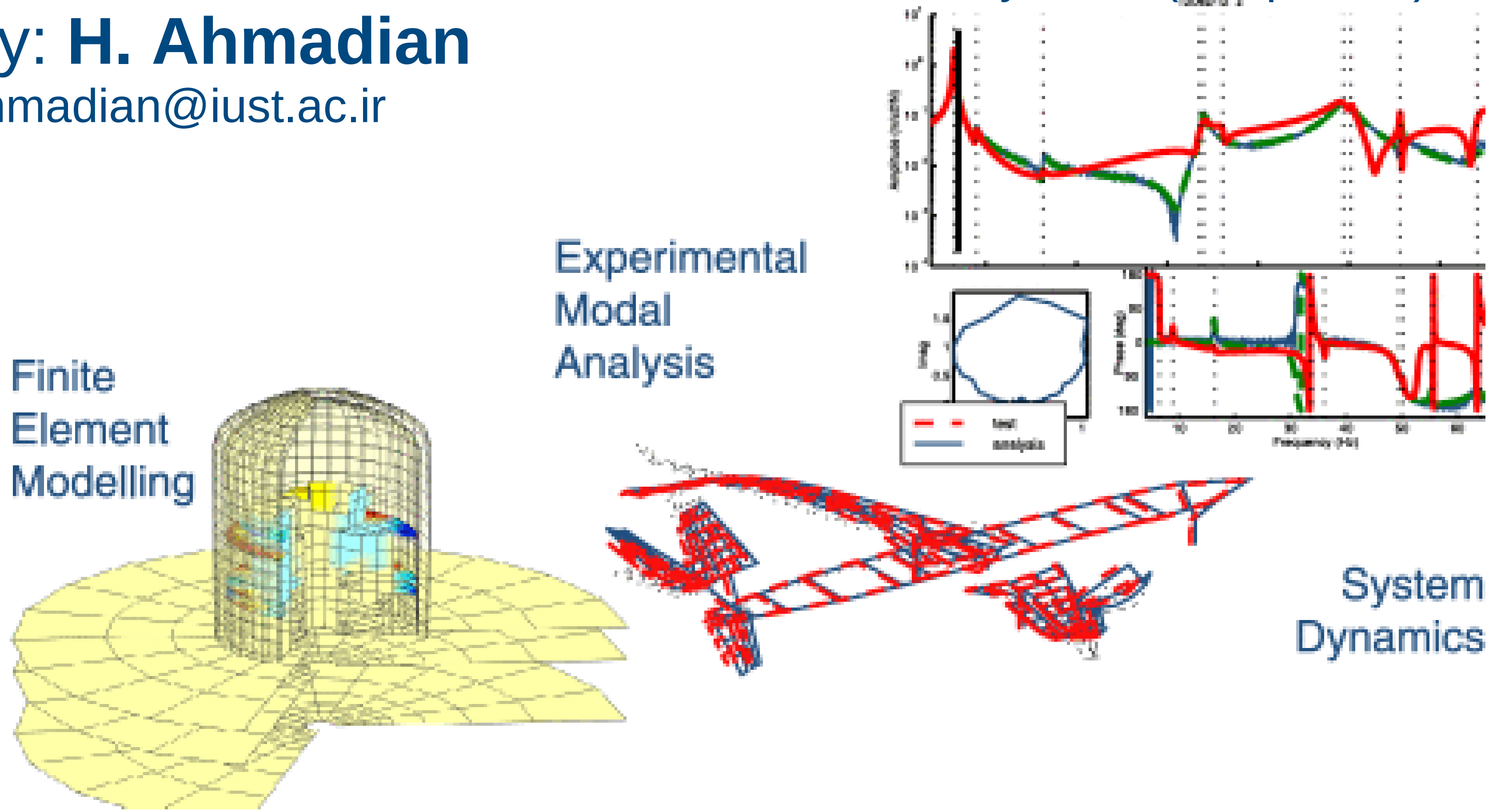

School of Mechanical Engineering

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Mathematical Models of Continuous Systems

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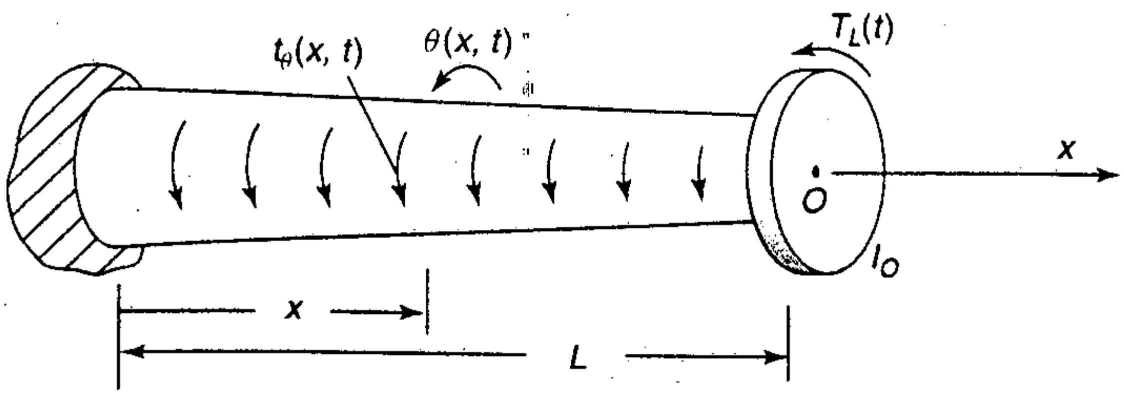
APPLICATION 'OF HAMILTON'S PRINCIPLE: TORSION OF A CIRCULAR ROD

$$\mathcal{V} = \frac{1}{2}\int_0^L GJ(x)[\theta'(x,t)]^2\,dx$$

$$\mathcal{T} = \mathcal{T}_{\text{rod}} + \mathcal{T}_{\text{pulley}} = \frac{1}{2}\int_0^L \rho I_p(x)[\dot{\theta}(x,t)]^2\,dx + \frac{1}{2}I_O[\dot{\theta}(L,t)]^2$$

$$\delta\mathcal{W}_{\text{nc}} = \int_0^L t_\theta(x,t)\,\delta\theta(x,t)\,dx + T_L\,\delta\theta(L,t)$$



APPLICATION 'OF HAMILTON'S PRINCIPLE: TORSION OF A CIRCULAR ROD

$$\boxed{\int_{t_1}^{t_2} \delta(\mathcal{T} - \mathcal{V})\, dt + \int_{t_1}^{t_2} \delta \mathcal{W}_{nc}\, dt = 0}$$

$$\int_{t_1}^{t_2} \delta\left[\frac{1}{2}\int_0^L \rho I_p(\dot{\theta})^2\, dx + \frac{1}{2} I_O[\dot{\theta}(L, t)]^2 - \frac{1}{2}\int_0^L GJ(\theta')^2\, dx\right] dt$$

$$+ \int_{t_1}^{t_2} \left[\int_0^L t_\theta(x, t)\, \delta\theta(x, t)\, dx + T_L\, \delta\theta(L, t)\right] dt = 0$$

APPLICATION 'OF HAMILTON'S PRINCIPLE: TORSION OF A CIRCULAR ROD

$$\delta \mathcal{V} = \int_0^L GJ\theta' \, \delta\theta' \, dx$$

$$\int_{t_1}^{t_2} \delta \mathcal{V} \, dt = \int_{t_1}^{t_2} \left(\int_0^L GJ\theta' \, \delta\theta' \, dx \right) dt$$

$$= \int_{t_1}^{t_2} \left[(GJ\theta') \, \delta\theta \Big|_0^L - \int_0^L (GJ\theta')' \, \delta\theta \, dx \right] dt$$

APPLICATION 'OF HAMILTON'S PRINCIPLE: TORSION OF A CIRCULAR ROD

$$\delta T = \int_0^L \rho I_p \dot{\theta}\, \delta\dot{\theta}\, dx + I_O \dot{\theta}(L,t)\, \delta\dot{\theta}(L,t)$$

$$\int_{t_1}^{t_2} \delta T dt = \int_0^L \left(\int_{t_1}^{t_2} \rho I_p \dot{\theta}\, \delta\dot{\theta}\, dt \right) dx + \int_{t_1}^{t_2} I_O \dot{\theta}(L,t)\, \delta\dot{\theta}(L,t)\, dt$$

$$= \int_0^L \left[(\rho I_p \dot{\theta}\, \delta\theta)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (\rho I_p \ddot{\theta}\, \delta\theta)\, dt \right] dx$$

$$+ [I_O \dot{\theta}(L,t)\, \delta\theta(L,t)]\Big|_{t_1}^{t_2} - I_O \ddot{\theta}(L,t)\, \delta\theta(L,t)$$

APPLICATION 'OF HAMILTON'S PRINCIPLE: TORSION OF A CIRCULAR ROD

$$\int_{t_1}^{t_2} \int_0^L [(GJ\theta')' - \rho I_p \ddot{\theta} + t_\theta(x,t)]\delta\theta \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \{[(GJ\theta')\,\delta\theta]_{x=0} - [(GJ\theta' - T_L)\,\delta\theta]_{x=L} - I_O \ddot{\theta}(L,t)\,\delta\theta(L,t)\} \, dt = 0$$

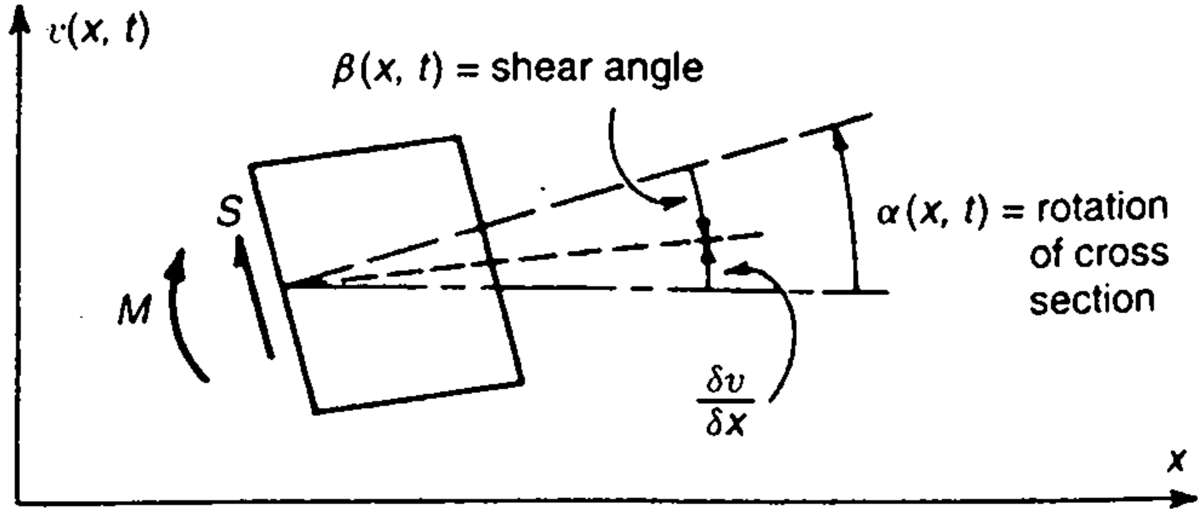
$$\boxed{-(GJ\theta')' + \rho I_p \ddot{\theta} = t_\theta(x,t), \qquad 0 < x < L}$$

$$\boxed{(GJ\theta')_{x=L} + I_O \ddot{\theta}(L,t) = T_L(t)}$$

$$\delta\theta(0,t) = 0,$$

BEAM FLEXURE INCLUDING SHEAR DEFORMATION AND ROTATORY INERTIA

$$\beta = \alpha - \frac{\partial v}{\partial x}$$



$$\mathcal{V}_b = \frac{1}{2}\int_0^L EI\,(\alpha')^2\,dx$$

$$\mathcal{V}_s = \frac{1}{2}\int_0^L \kappa G A \beta^2\,dx$$

$$\mathcal{V}_s = \frac{1}{2}\int_0^L \left(\iint_A \tau\gamma\, dA\right) dx$$

$$\kappa = \frac{5}{6}$$

BEAM FLEXURE INCLUDING SHEAR DEFORMATION AND ROTATORY INERTIA

$$\mathcal{V}_b = \frac{1}{2}\int_0^L EI(\alpha')^2\,dx \qquad \mathcal{V}_s = \frac{1}{2}\int_0^L \kappa GA\beta^2\,dx$$

$$\mathcal{T} = \frac{1}{2}\int_0^L \rho A(\dot{v})^2\,dx + \frac{1}{2}\int_0^L \rho I(\dot{\alpha})^2\,dx$$

$$\delta\mathcal{W}_{nc} = \int_0^L p_y(x,t)\,\delta v(x,t)\,dx$$

Application of the Extended Hamilton's Principle

$$\boxed{\int_{t_1}^{t_2} \delta(\mathcal{T} - \mathcal{V})\, dt + \int_{t_1}^{t_2} \delta \mathcal{W}_{nc}\, dt = 0}$$

$$\frac{1}{2}\int_{t_1}^{t_2}\int_0^L \delta\left[\rho A(\dot{v})^2 + \rho I(\dot{\alpha})^2 - EI(\alpha')^2 - \kappa GA(\alpha - v')^2\right] dx\, dt$$

$$+ \int_{t_1}^{t_2}\int_0^L p_y\, \delta v\, dx\, dt = 0$$

Application of the Extended Hamilton's Principle

Integrating by parts, and noting that $\delta v(x, t_1) = \delta v(x, t_2) = \delta\alpha(x, t_1) = \delta\alpha(x, t_2) = 0$, we obtain

$$\int_{t_1}^{t_2} \int_0^L \{-\rho A\ddot{v} - [\kappa GA(\alpha - v')]' + p_y\}\, \delta v\, dx\, dt$$

$$+ \int_{t_1}^{t_2} \int_0^L [-\rho I\ddot{\alpha} + (EI\alpha')' - \kappa GA(\alpha - v')]\, \delta\alpha\, dx\, dt$$

$$+ \int_{t_1}^{t_2} [\kappa GA(\alpha - v')\, \delta v\,]\Big|_0^L dt - \int_{t_1}^{t_2} [(EI\alpha')\, \delta\alpha\,]\Big|_0^L dt = 0$$

Timoshenko Beam Theory

$$\boxed{\begin{aligned} [\kappa GA(\alpha - v')]' + \rho A\ddot{v} &= p_y(x,t) \\ \kappa GA(\alpha - v') - (EI\alpha')' + \rho I\ddot{\alpha} &= 0 \end{aligned}}$$

$$\boxed{\begin{array}{ll} (\kappa\, GA\,\beta)\,\delta v = 0 & \text{at } x = 0 \\ (\kappa\, GA\,\beta)\,\delta v = 0 & \text{at } x = L \\ (EI\,\alpha')\,\delta\alpha = 0 & \text{at } x = 0 \\ (EI\,\alpha')\,\delta\alpha = 0 & \text{at } x = L \end{array}}$$

Timoshenko Beam Theory

If the beam has uniform cross-sectional properties, the two coupled PDEs may be combined to give a single equation in v:

$$\alpha' = v'' + \frac{1}{\kappa GA}(p_y - \rho A \ddot{v})$$

$$\underbrace{EI\frac{\partial^4 v}{\partial x^4} - \left(p_y - \rho A \frac{\partial^2 v}{\partial t^2}\right)}_{\text{Bernoulli–Euler theory}} \underbrace{-\rho I \frac{\partial^4 v}{\partial x^2 \partial t^2}}_{\text{principal rotatory inertia term}}$$

$$\underbrace{+\frac{EI}{\kappa GA}\frac{\partial^2}{\partial x^2}\left(p_y - \rho A \frac{\partial^2 v}{\partial t^2}\right)}_{\text{principal shear deformation term}} \underbrace{-\frac{\rho I}{\kappa GA}\frac{\partial^2}{\partial t^2}\left(p_y - \rho A \frac{\partial^2 v}{\partial t^2}\right)}_{\text{combined rotatory inertia and shear deformation}} = 0$$

Mathematical Models of Continuous Systems

- Applications of Newton's Laws: Axial Deformation and Torsion
- Application of Newton's Laws: Transverse Vibration of Linearly Elastic Beams (Bernoulli-Euler Beam Theory)
- Application of Hamilton's Principle: Torsion of a Rod with Circular Cross Section
- Application of the Extended Hamilton's Principle: Beam Flexure Including Shear Deformation and Rotatory Inertia (Timoshenko Beam Theory)

Structural Dynamics

Lecture 15: Free Vibration of Continuous Systems (Chapter 13)

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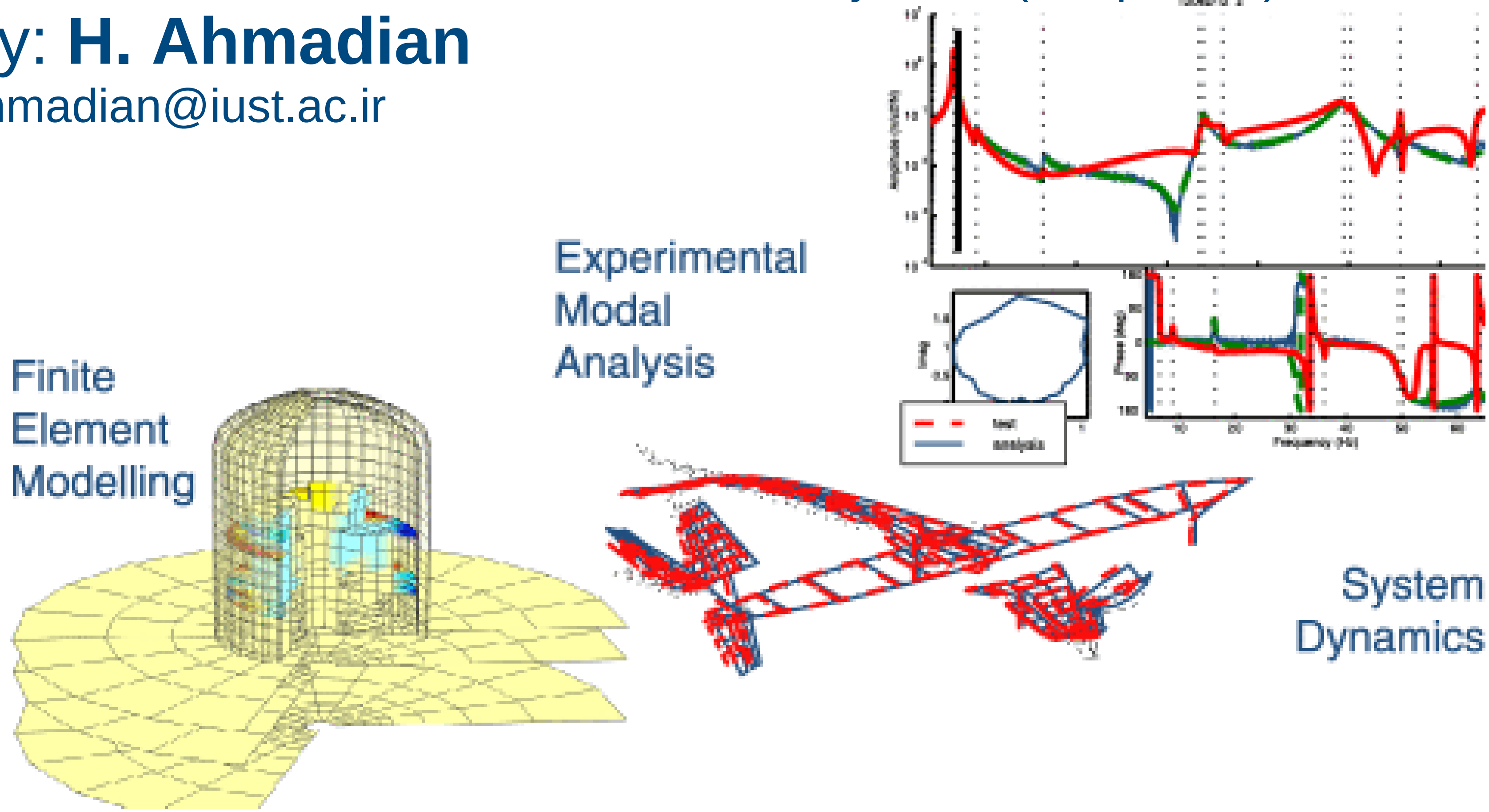

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Free Vibration of Continuous Systems

- Free Axial and Torsional Vibration
- Free Transverse Vibration of Bernoulli-Euler Beams
- Rayleigh's Method for Approximating the Fundamental Frequency of a Continuous System
- Free Transverse Vibration of Beams Including Shear Deformation and Rotatory Inertia
- Some Properties of Natural Modes of Continuous Systems
- Free Vibration of Thin Flat Plates

FREE AXIAL VIBRATION

$$(AEu')' - \rho A\ddot{u} = 0$$

$$u(x,t) = U(x)\cos(\omega t - \alpha)$$

$$(AEU')' + \omega^2(\rho AU) = 0$$

$$\frac{d^2U}{dx^2} + \frac{\rho\omega^2}{E}U = 0$$

FREE AXIAL VIBRATION

$$\frac{d^2U}{dx^2} + \lambda^2 U = 0 \qquad \lambda = \omega\sqrt{\frac{\rho}{E}}$$

$$U(x) = A_1 \cos\lambda x + A_2 \sin\lambda x$$

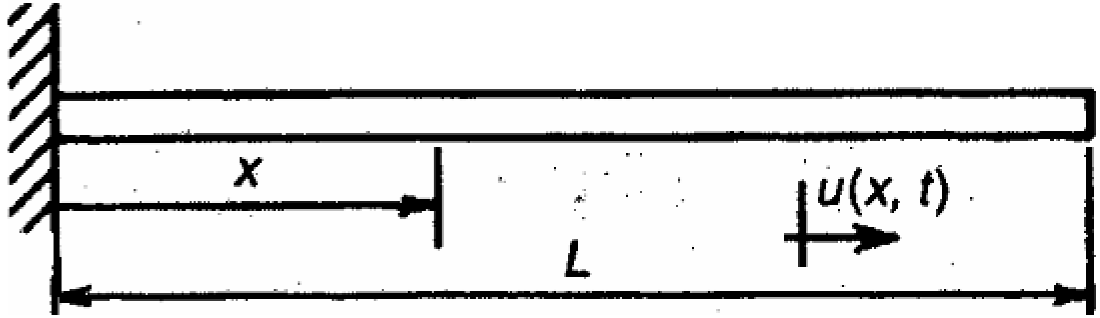
End conditions

Fixed end: $U = 0$ Free end: $\frac{dU}{dx} = 0$

FREE AXIAL VIBRATION

Example:

$$U(0) = \left.\frac{dU}{dx}\right|_{x=L} = 0$$



$$U(x) = A_1 \cos \lambda x + A_2 \sin \lambda x$$

$$U(0) = A_1 = 0 \qquad \left.\frac{dU}{dx}\right|_{x=L} = A_2 \lambda \cos \lambda L = 0 \qquad \cos \lambda L = 0$$

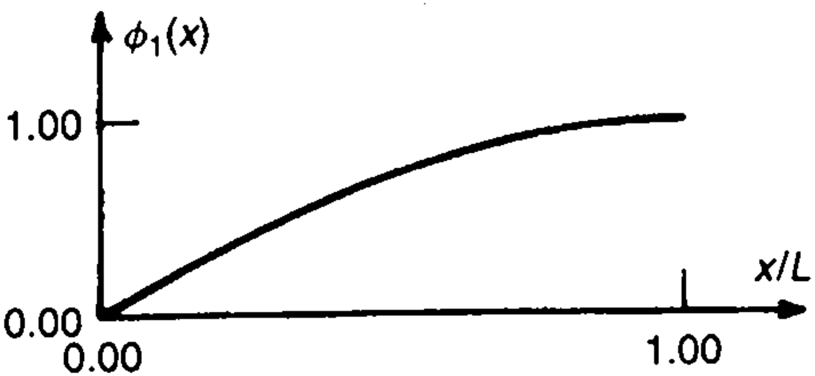
$$\lambda L = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, \left(r - \frac{1}{2}\right)\pi, \ldots$$

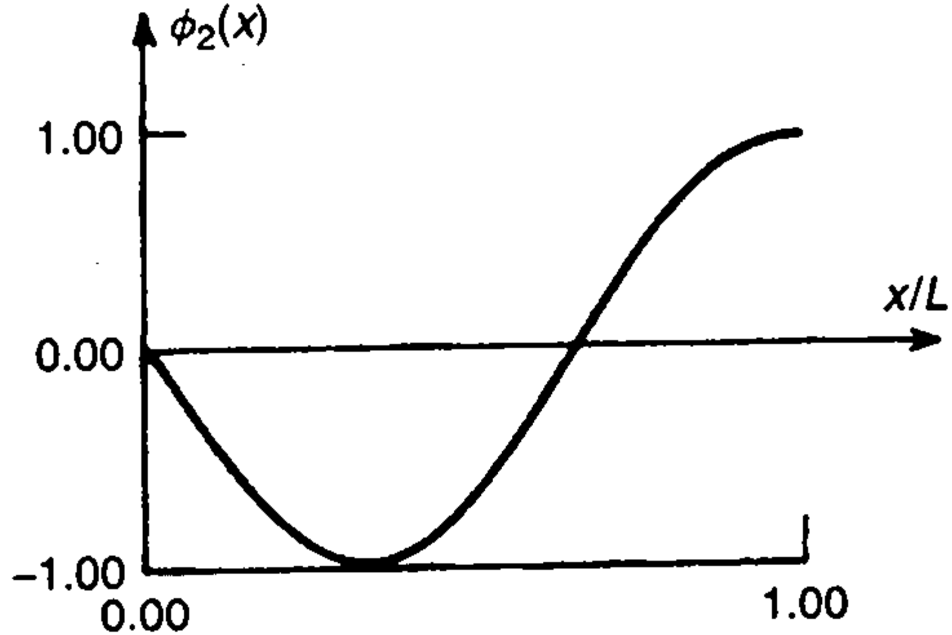
$$\omega_r = \lambda_r \left(\frac{E}{\rho}\right)^{1/2} = \frac{(\lambda L)_r}{L}\left(\frac{E}{\rho}\right)^{1/2} = \frac{(2r-1)\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$$

FREE AXIAL VIBRATION

Example:

$$\phi_r(x) = \sin\left(\frac{2r-1}{2}\,\frac{\pi x}{L}\right)$$





FREE TRANSVERSE VIBRATION OF BERNOULLI-EULER BEAMS

$$(EIV'')'' + \rho A \ddot{v} = 0$$

$$v(x,t) = V(x)\cos(\omega t - \alpha)$$

$$(EIV'')'' - \rho A \omega^2 V = 0$$

Free vibration of a uniform beam,

$$\frac{d^4 V}{dx^4} - \lambda^4 V = 0 \qquad \lambda^4 = \omega^2 \frac{\rho A}{EI}$$

$$V(x) = A_1 e^{\lambda x} + A_2 e^{-\lambda x} + A_3 e^{i\lambda x} + A_4 e^{-i\lambda x}$$

FREE TRANSVERSE VIBRATION OF BERNOULLI-EULER BEAMS

$$V(x) = A_1 e^{\lambda x} + A_2 e^{-\lambda x} + A_3 e^{i\lambda x} + A_4 e^{-i\lambda x}$$

Two useful alternative forms are

$$V(x) = B_1 e^{\lambda x} + B_2 e^{-\lambda x} + B_3 \sin \lambda x + B_4 \cos \lambda x$$

and

$$V(x) = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x + C_4 \cos \lambda x$$

There are five constants in the general solution: the four amplitude constants and the eigenvalue.

FREE TRANSVERSE VIBRATION OF BERNOULLI-EULER BEAMS: Example

$$V(x) = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x + C_4 \cos \lambda x$$

$$V(0) = 0, \qquad \left.\frac{d^2 V}{dx^2}\right|_{x=0} = 0 \longrightarrow C_2 = C_4 = 0.$$

$$V(L) = 0, \qquad \left.\frac{d^2 V}{dx^2}\right|_{x=L} = 0 \longrightarrow \begin{aligned} & C_1 \sinh \lambda L + C_3 \sin \lambda L = 0 \\ & (C_1 \sinh \lambda L - C_3 \sin \lambda L) = 0 \end{aligned}$$

$$C_1 = 0 \qquad \sin \lambda L = 0 \qquad \lambda_r = r\pi / L, \qquad \omega_r = \left(\frac{r\pi}{L}\right)^2 \left(\frac{EI}{\rho A}\right)^{1/2}$$

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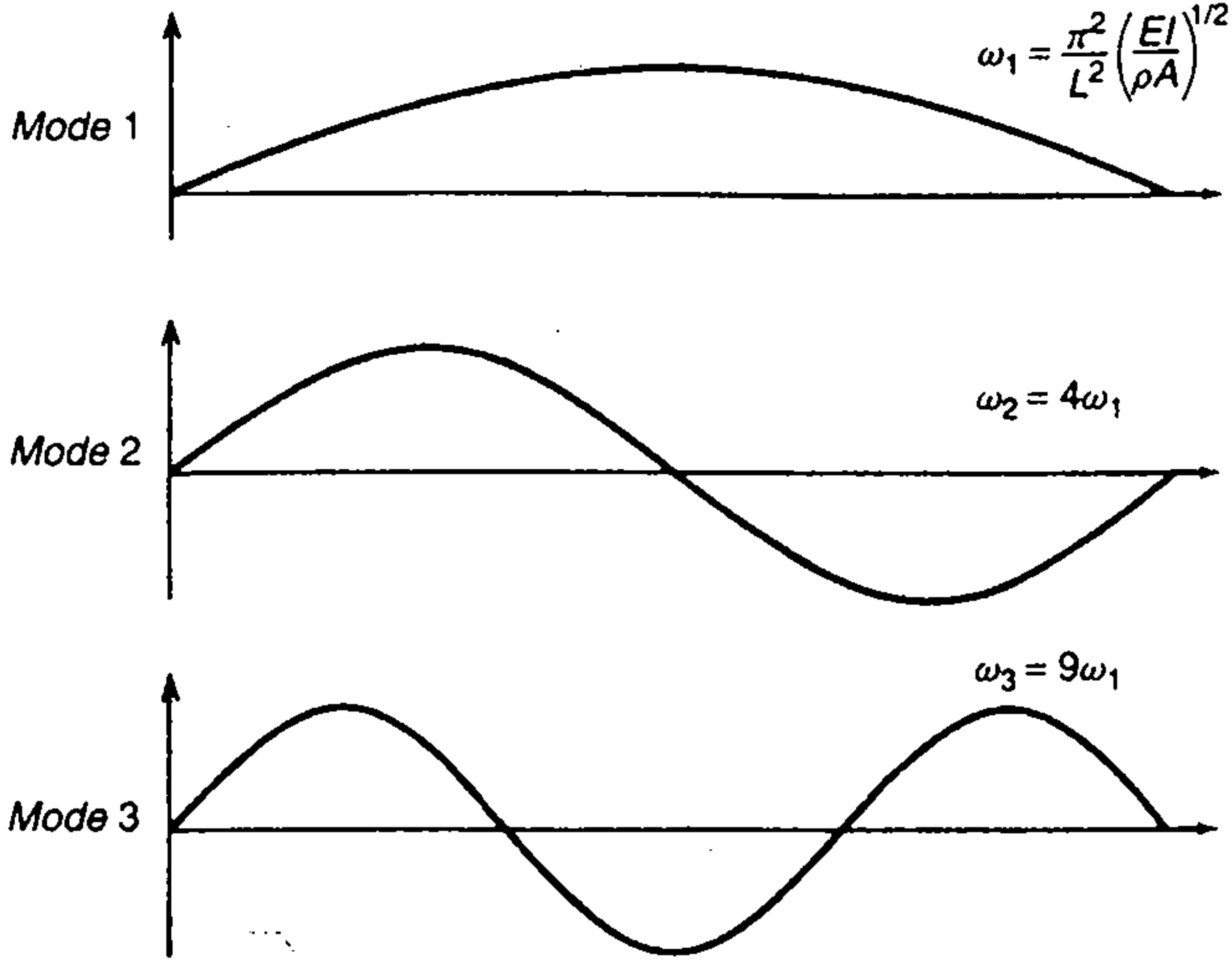
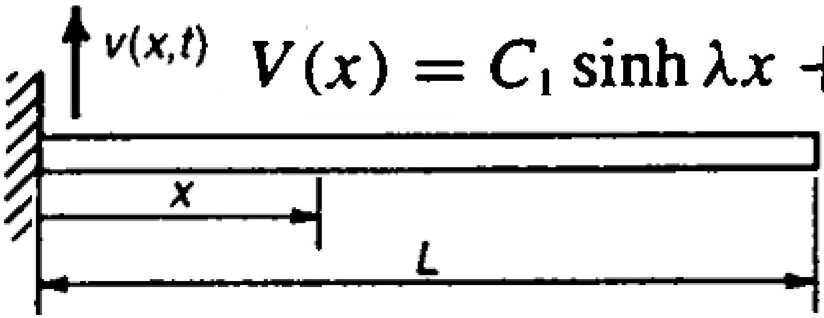
FREE TRANSVERSE VIBRATION OF BERNOULLI-EULER BEAMS: Example

$$\phi_r(x) = \sin\frac{r\pi x}{L}$$



FREE TRANSVERSE VIBRATION OF BERNOULLI-EULER BEAMS: Example



$$V(x) = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x + C_4 \cos \lambda x$$

$$V(0) = 0, \quad \left.\frac{dV}{dx}\right|_{x=0} = 0 \qquad \left.\frac{d^2V}{dx^2}\right|_{x=L} = 0, \quad \left.\frac{d^3V}{dx^3}\right|_{x=L} = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & \lambda & 0 \\ \lambda^2 \sinh \lambda L & \lambda^2 \cosh \lambda L & -\lambda^2 \sin \lambda L & -\lambda^2 \cos \lambda L \\ \lambda^3 \cosh \lambda L & \lambda^3 \sinh \lambda L & -\lambda^3 \cos \lambda L & \lambda^3 \sin \lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

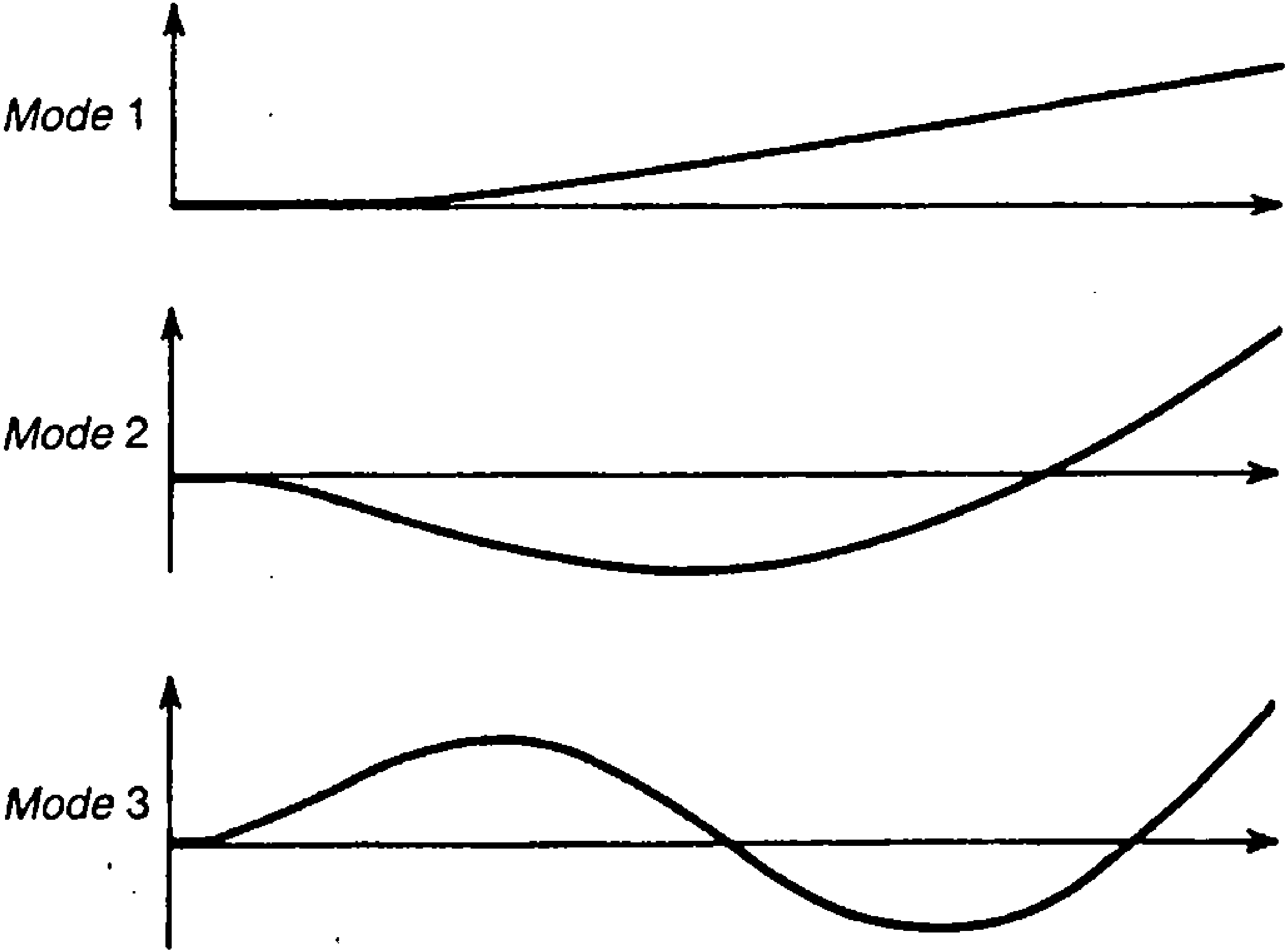
FREE TRANSVERSE VIBRATION OF BERNOULLI-EULER BEAMS: Example

$$\cos \lambda L \cosh \lambda L + 1 = 0$$

$$\lambda_1 L = 1.8751, \qquad \lambda_2 L = 4.6941,$$

$$\lambda_3 L = 7.8548, \qquad \lambda_4 L = 10.996$$

$$\omega_r = \frac{(\lambda_r L)^2}{L^2} \left(\frac{EI}{\rho A} \right)^{1/2}$$


Mode 1
Mode 2
Mode 3

RAYLEIGH'S METHOD FOR APPROXIMATING THE FUNDAMENTAL FREQUENCY OF A CONTINUOUS SYSTEM

Lord Rayleigh observed that for undamped free vibration, the motion is simple harmonic motion. Thus,

$$v(x,t) = V(x)\cos\omega_R t = C\psi(x)\cos\omega_R t$$

Rayleigh also observed that energy is conserved.

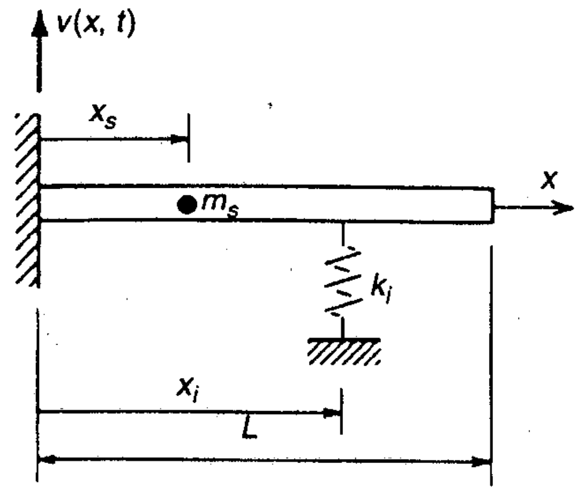
- the maximum kinetic energy is equal to the maximum potential energy, that is,

$$T_{\max} = V_{\max}$$

RAYLEIGH'S METHOD FOR APPROXIMATING THE FUNDAMENTAL FREQUENCY OF A CONTINUOUS SYSTEM

$$\mathcal{V} = \frac{1}{2}\int_0^L EI\,(v'')^2\,dx + \frac{1}{2}k_i\,v_i^2$$

$$\mathcal{T} = \frac{1}{2}\int_0^L \rho A(\dot{v})^2\,dx + \frac{1}{2}m_s\dot{v}_s^2$$



$$v(x,t) = C\psi(x)\cos\omega_R t$$

$$\mathcal{V}_{\max} = \frac{1}{2}kC^2 \qquad k = \int_0^L EI\,(\psi'')^2\,dx + k_i[\psi(x_i)]^2$$

$$\mathcal{T}_{\max} = \frac{1}{2}\omega_R^2 mC^2 \qquad m = \int_0^L \rho A\psi^2\,dx + m_s[\psi(x_s)]^2$$

$$\boxed{\mathcal{R}(V) = \omega_R^2 = \frac{k}{m}}$$

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Example : Approximate fundamental frequency of a uniform cantilever beam

The shape function $\longrightarrow$ $$\psi(x) = \left(\frac{x}{L}\right)^2$$

$$m = \int_0^L \rho A \psi^2 \, dx = \frac{\rho A L}{5}$$

$$\psi''(x) = \frac{2}{L^2} \longrightarrow k = \int_0^L EI (\psi'')^2 \, dx = \frac{4EI}{L^3}$$

$$\omega_R^2 = \frac{k}{m} = \frac{20EI}{\rho A L^4} \longrightarrow \omega_R = \frac{4.472}{L^2}\left(\frac{EI}{\rho A}\right)^{1/2}$$

$$\omega_1 = \frac{3.516}{L^2}\left(\frac{EI}{\rho A}\right)^{1/2}$$

FREE TRANSVERSE VIBRATION OF BEAMS INCLUDING SHEAR DEFORMATION AND ROTATORY INERTIA

Consider a uniform, simply supported beam:

Equations of motion:

$$\alpha' = v'' - \frac{\rho}{\kappa G}\ddot{v}$$

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} - \rho A r_G^2\left(1 + \frac{E}{\kappa G}\right)\frac{\partial^4 v}{\partial x^2\,\partial t^2} + \frac{\rho^2 A r_G^2}{\kappa G}\frac{\partial^4 v}{\partial t^4} = 0$$

Geometric boundary conditions:

$$v(0,t) = v(L,t) = 0$$

Natural boundary conditions:

$$\alpha'(0,t) = \alpha'(L,t) = 0$$

FREE TRANSVERSE VIBRATION OF BEAMS INCLUDING SHEAR DEFORMATION AND ROTATORY INERTIA

$$v(x,t) = V(x)\cos\omega t$$

$$\alpha' = \left(V'' - \frac{\rho\omega^2}{\kappa G}V\right)\cos\omega t$$

So the boundary conditions reduce, respectively, to

$$V(0) = V(L) = 0$$

$$V''(0) = V''(L) = 0$$

FREE TRANSVERSE VIBRATION OF BEAMS INCLUDING SHEAR DEFORMATION AND ROTATORY INERTIA

$$v(x,t) = V(x)\cos\omega t$$

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} - \rho A r_G^2\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 v}{\partial x^2\,\partial t^2} + \frac{\rho^2 A r_G^2}{\kappa G}\frac{\partial^4 v}{\partial t^4} = 0$$

$$\Downarrow$$

$$V^{iv} - \lambda^4 V + \lambda^4 r_G^2\left(1+\frac{E}{\kappa G}\right)V'' + \lambda^8 r_G^4\frac{E}{\kappa G}V = 0$$

The simply supported beam mode shape satisfies both the boundary conditions, and the equation of motion:

$$V_r(x) = C\sin\frac{r\pi x}{L}$$

FREE TRANSVERSE VIBRATION OF BEAMS INCLUDING SHEAR DEFORMATION AND ROTATORY INERTIA

$$\underbrace{\left(\frac{r\pi}{L}\right)^4 - \lambda^4}_{(a)} + \underbrace{\lambda^4 r_G^2 \left(\frac{r\pi}{L}\right)^2}_{(b)} + \underbrace{\lambda^4 r_G^2 \frac{E}{\kappa G}\left(\frac{r\pi}{L}\right)^2}_{(c)} + \underbrace{\lambda^8 r_G^4 \frac{E}{\kappa G}}_{(d)} = 0$$

Bernoulli-Euler beam characteristic equation

The shear correction term

The rotatory inertia correction term

Free Vibration of Continuous Systems

- Free Axial and Torsional Vibration
- Free Transverse Vibration of Bernoulli-Euler Beams
- Rayleigh's Method for Approximating the Fundamental Frequency of a Continuous System
- Free Transverse Vibration of Beams Including Shear Deformation and Rotatory Inertia
- Some Properties of Natural Modes of Continuous Systems
- Free Vibration of Thin Flat Plates

Structural Dynamics

Lecture 16: Free Vibration of Continuous Systems (Chapter 13)

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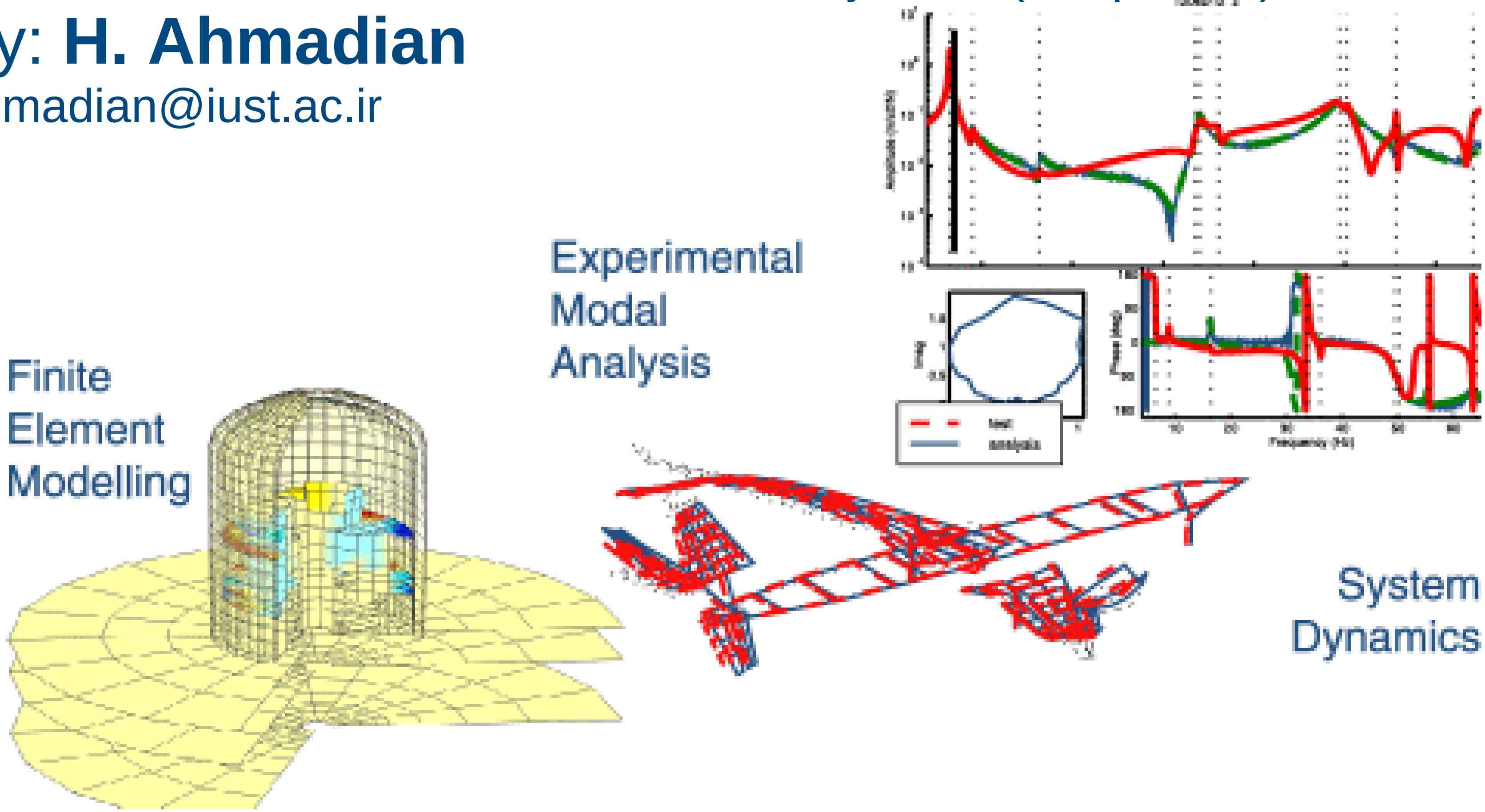

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Free Vibration of Continuous Systems

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SOME PROPERTIES OF NATURAL MODES OF CONTINUOUS SYSTEMS

Following properties associated with the modes are considered:

- scaling (or normalization),
- orthogonality,
- the expansion theorem, and
- the Rayleigh quotient.

These are illustrated by using the Bernoulli-Euler beam equations.

SOME PROPERTIES OF NATURAL MODES OF CONTINUOUS SYSTEMS: scaling

$$\boxed{M_r = \int_0^L \rho A \phi_r^2 \, dx}$$

$$\boxed{K_r = \int_0^L EI \, (\phi_r'')^2 \, dx}$$

$$\int_0^L (EI\,\phi_r'')''\phi_r \, dx - \omega_r^2 \int_0^L \rho A \phi_r^2 \, dx = 0$$

$$(EI\,\phi_r'')'\phi_r \Big|_0^L - (EI\,\phi_r'')\phi_r' \Big|_0^L + \int_0^L EI \, (\phi_r'')^2 \, dx - \omega_r^2 \int_0^L \rho A \phi_r^2 \, dx = 0$$

$$\boxed{\omega_r^2 = \frac{K_r}{M_r}}$$

Orthononnal modes

$$M_r = \int_0^L \rho A \phi_r^2 \, dx = 1$$

SOME PROPERTIES OF NATURAL MODES OF CONTINUOUS SYSTEMS: orthogonality

$$\int_0^L (EI\,\phi_r'')''\phi_s\,dx - \omega_r^2 \int_0^L \rho A \phi_r \phi_s\,dx = 0$$

$$\int_0^L EI\,(\phi_r'')\phi_s''\,dx - \omega_r^2 \int_0^L \rho A \phi_r \phi_s\,dx = 0$$

$$\int_0^L EI\,(\phi_s'')\phi_r''\,dx - \omega_s^2 \int_0^L \rho A \phi_r \phi_s\,dx = 0$$

$$(\omega_r^2 - \omega_s^2) \int_0^L \rho A \phi_r \phi_s\,dx = 0$$

SOME PROPERTIES OF NATURAL MODES OF CONTINUOUS SYSTEMS: orthogonality

$$\boxed{\int_0^L \rho A \phi_r \phi_s \, dx = 0, \qquad \omega_r \neq \omega_s}$$

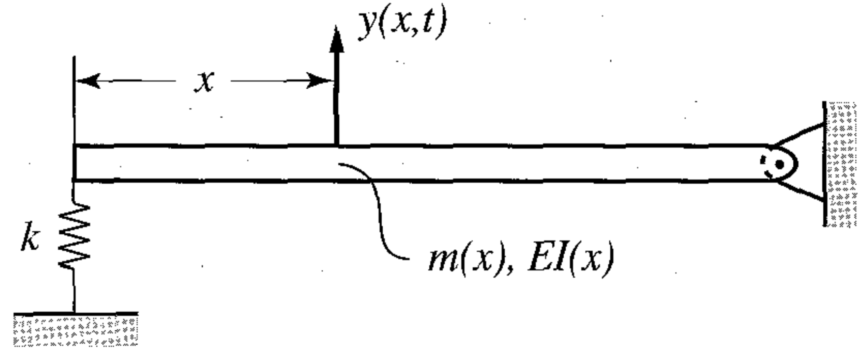
$$\boxed{\int_0^L EI \phi_r'' \phi_s'' \, dx = 0, \qquad \omega_r \neq \omega_s}$$

SOME PROPERTIES OF NATURAL MODES OF CONTINUOUS SYSTEMS: orthogonality

To demonstrate the orthogonality relations for beams with loaded boundaries, we consider two distinct solutions of the eigenvalue problem:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]=\omega_r^2 m(x)Y_r(x),\; 0<x<L$$

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_s(x)}{dx^2}\right]=\omega_s^2 m(x)Y_s(x),$$



$$Y(x)=0,\; EI(x)\frac{d^2Y(x)}{dx^2}=0,\; x=L$$

$$EI(x)\frac{d^2Y(x)}{dx^2}=0,\; -\frac{d}{dx}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]=kY(x),\; x=0$$

Orthogonality relations for beams

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx = \omega_r^2\int_0^L m(x)Y_s(x)Y_r(x)dx$$

$$\begin{aligned}\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx &= \left\{Y_s(x)\frac{d}{dx}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\right\}\Bigg|_0^L \\ &\quad -\left[\frac{dY_s(x)}{dx}EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\Bigg|_0^L \\ &\quad +\int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx \\ &= kY_s(0)Y_r(0)+\int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx\end{aligned}$$

$$kY_s(0)Y_r(0)+\int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx = \omega_r^2\int_0^L m(x)Y_s(x)Y_r(x)dx$$

Orthogonality relations for beams

$$kY_s(0)Y_r(0) + \int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx = \omega_r^2 \int_0^L m(x)Y_s(x)Y_r(x)dx$$

$$kY_r(0)Y_s(0) + \int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx = \omega_s^2 \int_0^L m(x)Y_r(x)Y_s(x)dx$$

$$\int_0^L m(x)Y_r(x)Y_s(x) = 0,\ r,s = 1,2,\ldots;\ \omega_r^2 \neq \omega_s^2$$

$$\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx + kY_r(0)Y_s(0) = 0,$$

Expansion Theorem:

Any function *V(x)* that satisfies the same boundary conditions as are satisfied by a given set of orthonormal modes, and is such that *(EI V″)″* is a continuous function, can be represented by an absolutely and uniformly convergent series of the form

$$\boxed{V(x) = \sum_{r=1}^{\infty} c_r \phi_r(x)}$$

$$c_r = \int_0^L \rho A V \phi_r \, dx \qquad M_r = 1$$

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RESPONSE TO INITIAL EXCITATIONS: Beams in Bending Vibration

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] = m(x)\frac{\partial^2 y(x,t)}{\partial t^2},\ 0 < x < L$$

$$-\sum_{r=1}^{\infty}\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 Y_r(x)}{dx^2}\right]\eta_r(t) = \sum_{r=1}^{\infty} m(x)Y_r(x)\frac{d^2\eta_r(t)}{dt^2},$$

$$-\sum_{r=1}^{\infty}\left\{\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 Y_r(x)}{dx^2}\right]dx\right\}\eta_r(t)$$

$$= \sum_{r=1}^{\infty}\left[\int_0^L m(x)Y_s(x)Y_r(x)dx\right]\frac{d^2\eta_r(t)}{dt^2}$$

RESPONSE TO INITIAL EXCITATIONS: Beams in Bending Vibration

To demonstrate that every one of the natural modes can be excited independently of the other modes we select the initials as:

$$y_0(x) = AY_p(x)$$

$$\eta_r(0) = A\int_0^L \rho(x)Y_r(x)Y_p(x)dx = \begin{cases} A \text{ for } r = p \\ 0 \text{ for } r = 1, 2, \ldots, p-1, p+1, \ldots \end{cases}$$

$$\eta_r(t) = \begin{cases} A\cos\omega_r t \text{ for } r = p \\ 0 \text{ for } r = 1, 2, \ldots, p-1, p+1, \ldots \end{cases}$$

$$y(x,t) = AY_p(x)\cos\omega_p t$$

RESPONSE TO INITIAL EXCITATIONS:
Response of systems with tip masses

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] = m(x)\frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < L$$

Boundary conditions

$$u(0,t) = 0$$

$$-EA(x)\frac{\partial u(x,t)}{\partial x} = M\frac{\partial^2 u(x,t)}{\partial t^2}, \quad x = L$$

Initial conditions

$$u(x,0) = u_0(x), \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = v_0(x)$$

RESPONSE TO INITIAL EXCITATIONS:
Response of systems with tip masses

$$u(x,t) = \sum_{r=1}^{\infty} U_r(x)\eta_r(t)$$

$$\sum_{r=1}^{\infty}\left\{\int_0^L U_s(x)\frac{d}{dx}\left[EA(x)\frac{dU_r(x)}{dx}\right]dx\right\}\eta_r(t) = \sum_{r=1}^{\infty}\left[\int_0^L m(x)U_s(x)U_r(x)dx\right]\ddot{\eta}_r(t),$$

$$\int_0^L m(x)U_s(x)U_s(x)dx = \delta_{rs} - MU_r(L)U_s(L),$$

$$\int_0^L U_s(x)\frac{d}{dx}\left[EA(x)\frac{dU_r(x)}{dx}\right]dx = \left[U_s(x)EA(x)\frac{dU_r(x)}{dx}\right]\Big|_{x=L} - \omega_r^2\delta_{rs},$$

Observing from boundary condition

$$\sum_{r=1}^{\infty}\left[MU_r(x)\ddot{\eta}_r(t) + EA(x)\frac{dU_r(x)}{dx}\eta_r(t)\right]\Big|_{x=L}$$

$$= \left[M\frac{\partial^2 u(x,t)}{\partial t^2} + EA(x)\frac{\partial u(x,t)}{\partial x}\right]\Big|_{x=L} = 0$$

RESPONSE TO INITIAL EXCITATIONS:
Response of systems with tip masses

$$\ddot{\eta}_s(t) + \omega_s^2 \eta_s(t) = 0, \quad s = 1, 2, \ldots$$

$$\eta_s(t) = \eta_s(0) \cos \omega_s t + \frac{\dot{\eta}_s(0)}{\omega_s} \sin \omega_s t,$$

$$u(x, 0) = \sum_{s=1}^{\infty} U_s(x) \eta_s(0) = u_0(x)$$

$$\eta_s(0) = \int_0^L m(x) U_s(x) u_0(x) dx + M U_s(L) u_0(L),$$

Similarly,

$$\dot{\eta}_s(0) = \int_0^L m(x) U_s(x) v_0(x) dx + M U_s(L) v_0(L),$$

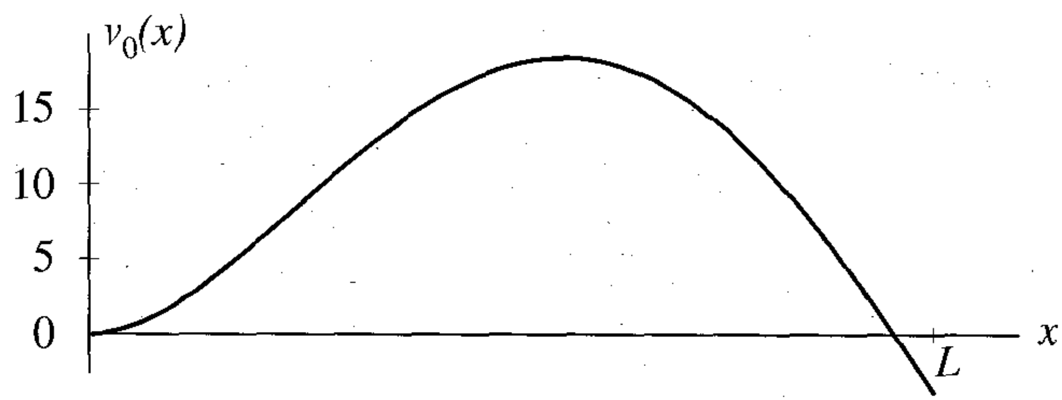
Example:

Response of a cantilever beam with a lumped mass at the end to the initial velocity:

$$-EI\frac{\partial^4 y(x,t)}{\partial x^4} = m\frac{\partial^2 y(x,t)}{\partial t^2},\ 0 < x < L$$

$$y(x,t) = 0,\ \frac{\partial y(x,t)}{\partial x} = 0,\ x = 0 \qquad EI\frac{\partial^2 y(x,t)}{\partial x^2} = 0,\ EI\frac{\partial^3 y(x,t)}{\partial x^3} = M\frac{\partial^2 y(x,t)}{\partial t^2},\ x = L$$

$$v_0(x) = 13.72\left(\frac{x}{L}\right)^2 - 23.22\left(\frac{x}{L}\right)^3 + 9.26\left(\frac{x}{L}\right)^4$$



Example:

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t)$$

$$m\int_0^L Y_r(x)Y_s(x)dx + MY_r(L)Y_s(L) = \delta_{rs},$$

$$EI\left\{\int_0^L Y_s(x)\frac{d^4Y_r(x)}{dx^4}dx - \left[Y_s(x)\frac{d^3Y_r(x)}{dx^3}\right]\Big|_{x=L}\right\} = \omega_r^2\delta_{rs},$$

$$\ddot{\eta}_s(t) + \omega_s^2\eta_s(t) - \sum_{r=1}^{\infty}\left\{Y_s(x)\left[MY_r(x)\ddot{\eta}_r(t) - EI\frac{d^3Y_r(x)}{dx^3}\eta_r(t)\right]\right\}\Big|_{x=L} = 0,$$

$$\ddot{\eta}_c + \omega_s^2\eta_s(t) = 0,\ \ s = 1,2\ldots \qquad \eta_s(t) = \frac{\dot{\eta}_s(0)}{\omega_s}\sin\omega_s t,$$

Example:

$$\dot{\eta}_s(0) = m\int_0^L Y_s(x)v_0(x)dx + MY_s(L)v_0(L)$$

$$= m\int_0^L Y_s(x)\left[13.72\left(\frac{x}{L}\right)^2 - 23.22\left(\frac{x}{L}\right)^3 + 9.26\left(\frac{x}{L}\right)^4\right]dx - 0.24MY_s(L),$$

$$M = mL,$$

$$y(x,t) = \sum_{r=1}^{\infty} C_r\left[\sin\beta_r x - \sinh\beta_r x - \frac{\sin\beta_r L + \sinh\beta_r L}{\cos\beta_r L + \cosh\beta_r L}(\cos\beta_r x - \cosh\beta_r x)\right]\sin\omega_r t$$

$$C_1 = -0.0404,\ C_2 = 0.7761,\ C_3 = -0.0003,$$

Because initial velocity resembles the 2nd mode

RESPONSE TO EXTERNAL EXCITATIONS

- The various types of distributed-parameter systems differ more in appearance than in vibrational characteristics.
- We consider the response of a beam in bending supported by a spring of stiffness ***k*** at *x=0* and pinned at *x=L*.

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},\ 0<x<L$$

$$EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}=0,\ \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+ky(x,t)=0,\ x=0 \qquad y(x,t)=0,\ EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}=0,\ x=L$$

RESPONSE TO EXTERNAL EXCITATIONS

$$y(x,t)=\sum_{r=1}^{\infty}Y_r(x)\eta_r(t)$$

Orthonormal modes

$$\int_0^L m(x)Y_r(x)Y_s(x)dx=\delta_{rs},\quad r,s=1,2,\ldots$$

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx=\omega_r^2\delta_{rs}$$

$$\ddot{\eta}_r(t)+\omega_r^2\eta_r(t)=N_r(t),$$

$$N_r(t)=\int_0^L Y_r(x)f(x,t)dx$$

RESPONSE TO EXTERNAL EXCITATIONS: **Harmonic Excitation**

$$f(x,t) = F(x)\cos\Omega t$$

$$N_r(t) = \left[\int_0^L Y_r(x)F(x)dx\right]\cos\Omega t = F_r\cos\Omega t,$$

$$F_r = \int_0^L Y_r(x)F(x)dx,\ r = 1, 2, \ldots$$

Controls which mode is excited.

$$\eta_r(t) = \frac{F_r}{\omega_r^2 - \Omega^2}\cos\Omega t,$$

Controls the resonance.

$$y(x,t) = \left[\sum_{r=1}^{\infty}\frac{F_r}{\omega_r^2 - \Omega^2}Y_r(x)\right]\cos\Omega t$$

RESPONSE TO EXTERNAL EXCITATIONS: **Arbitrary Excitation**

$$\eta_r(t) = \frac{1}{\omega_r} \int_0^t N_r(t-\tau) \sin \omega_r \tau \, d\tau, \quad r = 1, 2, \ldots$$

$$y(x,t) = \sum_{r=1}^{\infty} \frac{Y_r(x)}{\omega_r} \int_0^t N_r(t-\tau) \sin \omega_r \tau \, d\tau$$

The developments remain essentially the same for all other boundary conditions, and the same can be said about other systems.

Example

Derive the response of a uniform pinned-pinned beam to a concentrated force of amplitude F_0 acting at $x = L/2$ and having the form of a step function $f(x,t) = F_0 \delta(x - L/2) u(t)$

Orthonormal Modes

$$\omega_r = \beta_r^2 \sqrt{\frac{EI}{m}} = (r\pi)^2 \sqrt{\frac{EI}{mL^4}} \qquad Y_r(x) = \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L}, \; r = 1, 2, \ldots$$

$$N_r(t) = \int_0^L Y_r(x) f(x,t)\,dx = \sqrt{\frac{2}{mL}} F_0 u(t) \int_0^L \sin \frac{r\pi x}{L} \delta(x - L/2)\,dx$$

$$= \sqrt{\frac{2}{mL}} F_0 u(t) \sin \frac{r\pi}{2} = (-1)^{(r-1)/2} \sqrt{\frac{2}{mL}} F_0 u(t), \; r = \text{odd}$$

Example

$$\eta_r(t) = \frac{1}{\omega_r}\int_0^t N_r(t-\tau)\sin\omega_r\tau\,d\tau = \frac{(-1)^{(r-1)/2}F_0}{\omega_r}\sqrt{\frac{2}{mL}}\int_0^t u(t-\tau)\sin\omega_r\tau\,d\tau$$

$$= \frac{(-1)^{(r-1)/2}F_0}{\omega_r^2}\sqrt{\frac{2}{mL}}(1-\cos\omega_r t)$$

$$= \frac{(-1)^{(r-1)/2}F_0}{(r\pi)^4}\frac{mL^4}{EI}\sqrt{\frac{2}{mL}}\left[1-\cos(r\pi)^2\sqrt{\frac{EI}{mL^4}}t\right],\quad r = \text{odd}$$

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t) = \sum_{r=1,3,\ldots}^{\infty}\frac{(-1)^{(r-1)/2}F_0}{(r\pi)^4}\frac{mL^4}{EI}\frac{2}{mL}\sin\frac{r\pi x}{L}\left[1-\cos(r\pi)^2\sqrt{\frac{EI}{mL^4}}t\right]$$

$$= \frac{2F_0L^3}{\pi^4 EI}\sum_{r=1,3,\ldots}^{\infty}\frac{(-1)^{(r-1)/2}}{r^4}\sin\frac{r\pi x}{L}\left[1-\cos(r\pi)^2\sqrt{\frac{EI}{mL^4}}t\right]$$

Free Vibration of Continuous Systems

- Free Axial and Torsional Vibration
- Free Transverse Vibration of Bernoulli-Euler Beams
- Rayleigh's Method for Approximating the Fundamental Frequency of a Continuous System
- Free Transverse Vibration of Beams Including Shear Deformation and Rotatory Inertia
- Some Properties of Natural Modes of Continuous Systems
- Free Vibration of Thin Flat Plates

Structural Dynamics

Lecture 17: Free Vibration of Continuous Systems (Chapter 13)

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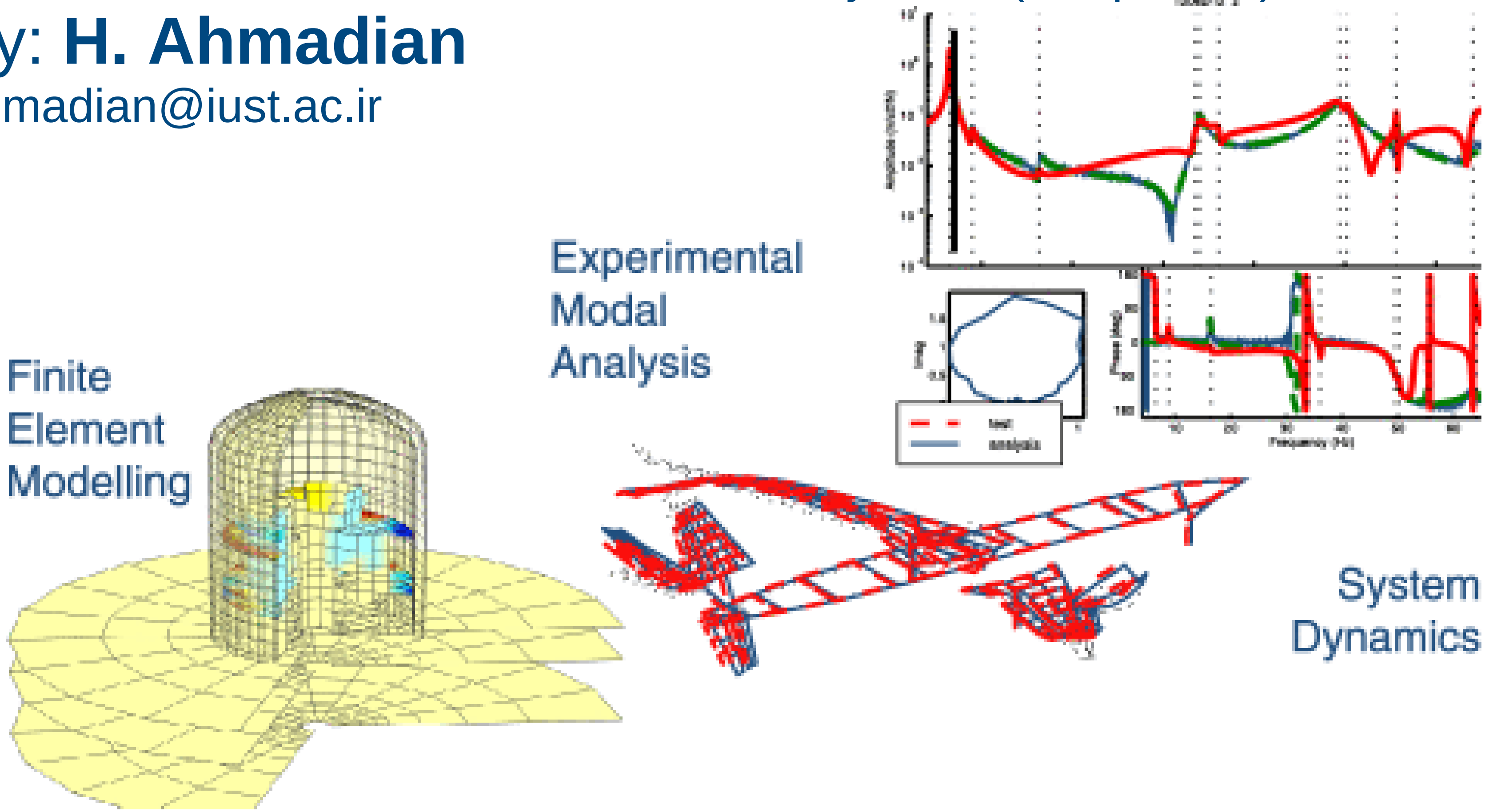

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Free Vibration of Continuous Systems

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Rayleigh Quotient:

$$\boxed{\mathcal{R}(V) = \frac{k}{m} = \frac{\int_0^L EI\,(V'')^2\,dx}{\int_0^L \rho A V^2\,dx}}$$

$$\mathcal{R}(V) = \frac{c_1^2\omega_1^2 + c_2^2\omega_2^2 + c_3^2\omega_3^2 + \cdots}{c_1^2 + c_2^2 + c_3^2 + \cdots}$$

$$\mathcal{R}(V) = \omega_1^2\,\frac{1 + (c_2/c_1)^2(\omega_2/\omega_1)^2 + (c_3/c_1)^2(\omega_3/\omega_1)^2 + \cdots}{1 + (c_2/c_1)^2 + (c_3/c_1)^2 + \cdots}$$

$$\boxed{\mathcal{R}(V) \geq \omega_1^2}$$

Example: **Lowest natural frequency of the fixed-free tapered rod in axial vibration**

$$m(x) = \frac{6m}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right], \quad EA(x) = \frac{6EA}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

The 1st mode of a uniform clamped-free rod as a trial function: $U(x) = \sin\frac{\pi x}{2L}$

A comparison function

$$R(U) = \omega^2 = \frac{\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^L m(x)U^2(x)dx} = \frac{EA}{m}\left(\frac{\pi}{2L}\right)^2 \frac{(L/12\pi^2)(5\pi^2+6)}{(L/12\pi^2)(5\pi^2-6)}$$

$$\omega = 1.7749\sqrt{\frac{EA}{mL^2}}$$

THE RAYLEIGH-RITZ METHOD

The method was developed by Ritz as an extension of Rayleigh's energy method.

- Although Rayleigh claimed that the method originated with him, the form in which the method is generally used is due to Ritz.

The first step in the Rayleigh-Ritz method is to construct the *minimizing sequence*:

undetermined coefficients

independent trial functions

$$Y^{(1)}(x) = a_1\phi_1(x)$$

$$Y^{(2)}(x) = a_1\phi_1(x) + a_2\phi_2(x) = \sum_{i=1}^{2} a_i\phi_i(x)$$

$$\vdots$$

$$Y^{(n)}(x) = a_1\phi_1(x) + a_2\phi_2(x) + \ldots + a_n\phi_n(x) = \sum_{i=1}^{n} a_i\phi_i(x)$$

THE RAYLEIGH-RITZ METHOD

$$\lambda^{(n)} = R(Y^{(n)}) = R(a_1, a_2, \ldots, a_n)$$

$$\delta R = \frac{\partial R}{\partial a_1}\delta a_1 + \frac{\partial R}{\partial a_2}\delta a_2 + \ldots + \frac{\partial R}{\partial a_n}\delta a_n = 0$$

The independence of the trial functions implies the independence of the coefficients, which in turn implies the independence of the variations

$$\delta a_1, \delta a_2, \ldots, \delta a_n \Longrightarrow$$

$$\frac{\partial R}{\partial a_i} = 0,\ i = 1, 2, \ldots, n$$

THE RAYLEIGH-RITZ METHOD

$$\lambda^{(n)} = R(a_1, a_2, \ldots, a_n) = \frac{N(a_1, a_2, \ldots, a_n)}{D(a_1, a_2, \ldots, a_n)}$$

$$\frac{\partial R}{\partial a_i} = \frac{(\partial N/\partial a_i)D - (\partial D/\partial a_i)N}{D^2} = \frac{1}{D}\left(\frac{\partial N}{\partial a_i} - \frac{N}{D}\frac{\partial D}{\partial a_i}\right)$$

$$= \frac{1}{D}\left(\frac{\partial N}{\partial a_i} - \lambda^{(n)}\frac{\partial D}{\partial a_i}\right) = 0, \; i = 1, 2, \ldots, n$$

$$\frac{\partial N}{\partial a_i} - \lambda^{(n)}\frac{\partial D}{\partial a_i} = 0, \; i = 1, 2, \ldots, n$$

Solving the equations amounts to determining the coefficients, as well as to determining $\lambda^{(n)}$

Example : **Solve the eigenvalue problem for the fixed-free tapered rod in axial vibration**

The comparison functions $\phi_i(x) = \sin\frac{(2i-1)\pi x}{2L}, \; i = 1, 2, \ldots, n$

$$V_{\max} = \frac{1}{2}\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx \qquad T_{\mathrm{ref}} = \frac{1}{2}\int_0^L m(x)U^2(x)dx$$

$$U^{(n)}(x) = \sum_{i=1}^{n} a_i^{(n)}\phi_i(x) = \sum_{i=1}^{n} a_i^{(n)} \sin\frac{(2i-1)\pi x}{2L}$$

$$V_{\max} \cong \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}^{(n)} a_i^{(n)} a_j^{(n)} \qquad T_{\mathrm{ref}} \cong \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}^{(n)} a_i^{(n)} a_j^{(n)}$$

Example :

$$k_{ij}^{(n)} = \int_0^L EA(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx$$

$$= \frac{6EA}{5}\frac{(2i-1)\pi}{2L}\frac{(2j-1)\pi}{2L}\int_0^L \left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\cos\frac{(2i-1)\pi x}{2L}\cos\frac{(2j-1)\pi x}{2L}dx,$$

$$m_{ij}^{(n)} = \int_0^L m(x)\phi_i(x)\phi_j(x)dx$$

$$= \frac{6m}{5}\int_0^L \left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\sin\frac{(2i-1)\pi x}{2L}\sin\frac{(2j-1)\pi x}{2L}dx,\ i,j=1,2,\dots,n$$

Example : *n* = 2

$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 1.383701 & 0.337500 \\ 0.337500 & 11.253305 \end{bmatrix} \qquad M^{(2)} = mL\begin{bmatrix} 0.439207 & 0.075991 \\ 0.075991 & 0.493245 \end{bmatrix}$$

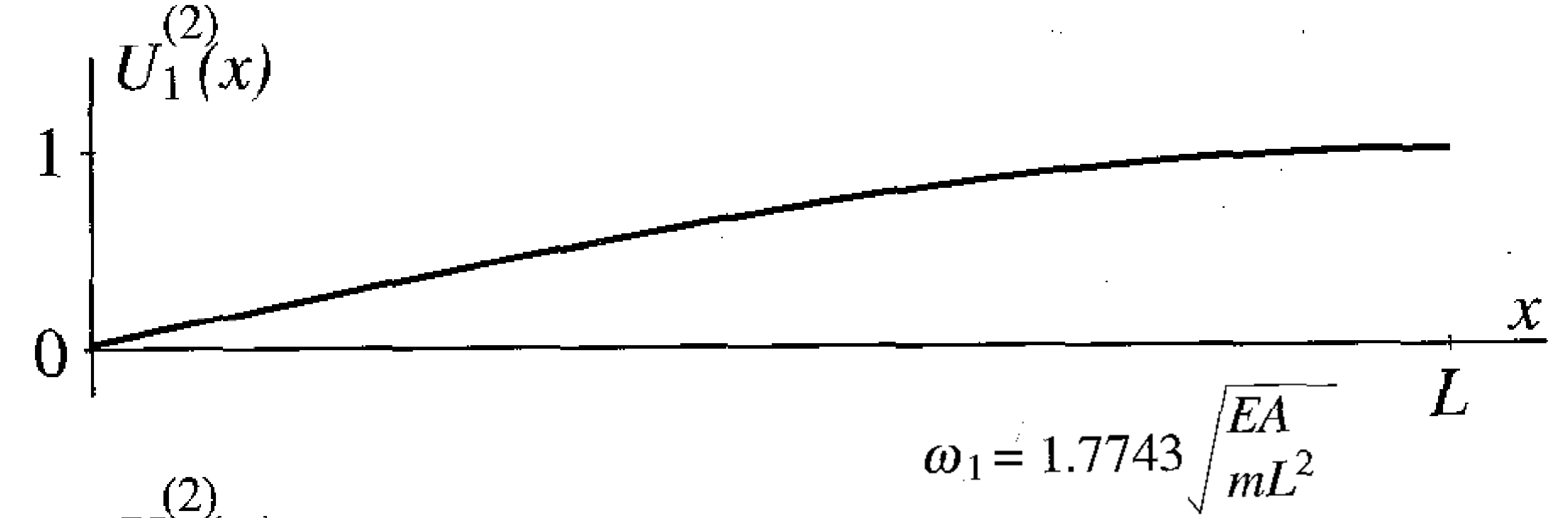
$$\omega_1^{(2)} = 1.774312\sqrt{\frac{EA}{mL^2}},\ \mathbf{a}_1^{(2)} = (mL)^{-1/2}\begin{bmatrix} 1.511481 \\ -0.015311 \end{bmatrix}$$

$$\omega_2^{(2)} = 4.825444\sqrt{\frac{EA}{mL^2}},\ \mathbf{a}_2^{(2)} = (mL)^{-1/2}\begin{bmatrix} -0.233683 \\ 1.443148 \end{bmatrix}$$

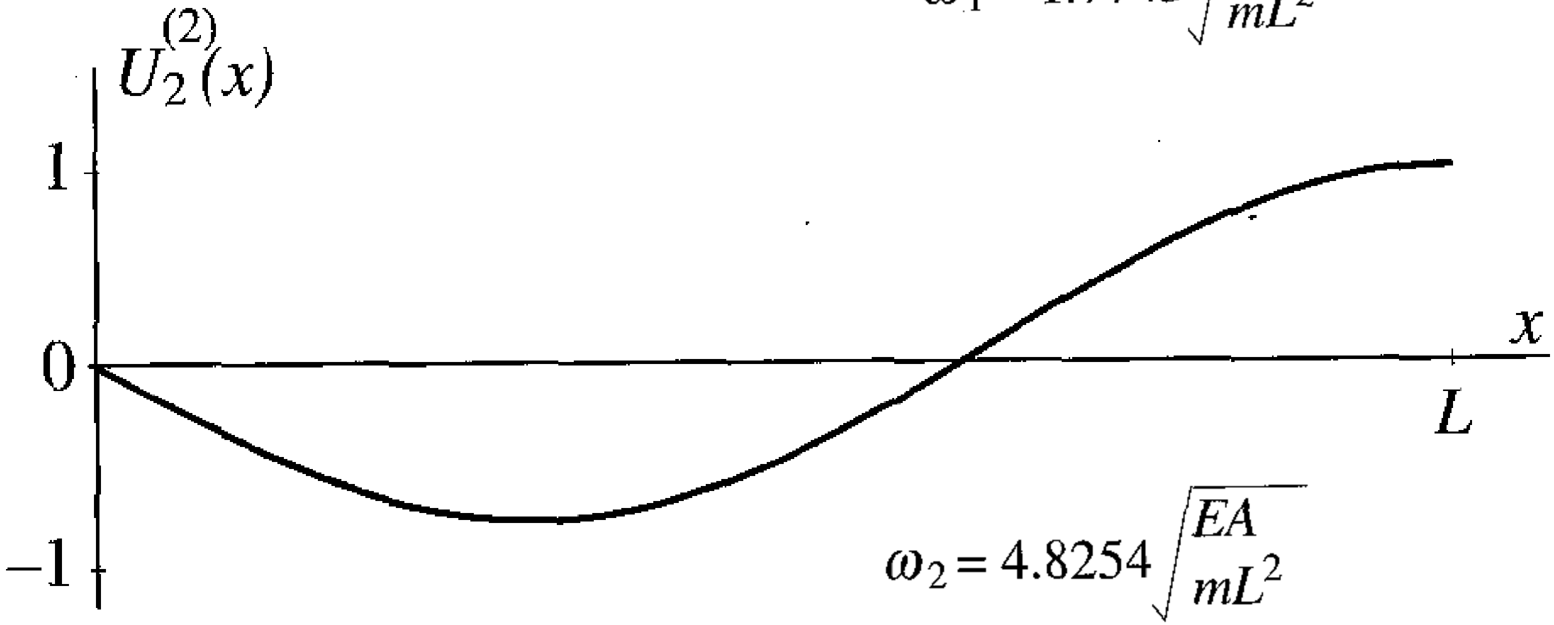
$$U_1^{(2)}(x) = 1.511481\sin\frac{\pi x}{2L} - 0.015311\sin\frac{3\pi x}{2L}$$

$$U_2^{(2)}(x) = -0.233683\sin\frac{\pi x}{2L} + 1.443148\sin\frac{3\pi x}{2L}$$

Example : $n = 2$



$U_1^{(2)}(x)$
1
0
x
L
$\omega_1 = 1.7743\sqrt{\frac{EA}{mL^2}}$



$U_2^{(2)}(x)$
1
0
-1
x
L
$\omega_2 = 4.8254\sqrt{\frac{EA}{mL^2}}$

Example : *n* = 3

$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} 1.383701 & 0.337500 & -0.104167 \\ 0.337500 & 11.253305 & 2.109375 \\ -0.104167 & 2.109375 & 30.992514 \end{bmatrix}$$

$$M^{(3)} = mL\begin{bmatrix} 0.439207 & 0.075991 & -0.021953 \\ 0.075991 & 0.493245 & 0.064592 \\ -0.021953 & 0.064592 & 0.497568 \end{bmatrix}$$

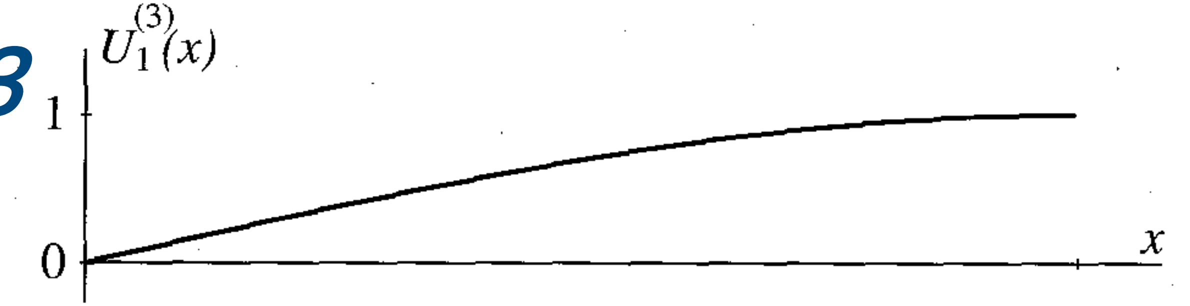
$$\omega_1^{(3)} = 1.774247\sqrt{\frac{EA}{mL^2}},\quad \mathbf{a}_1^{(3)} = (mL)^{-1/2}\begin{bmatrix} 1.511715 \\ -0.015872 \\ 0.002829 \end{bmatrix}$$

$$\omega_2^{(3)} = 4.822187\sqrt{\frac{EA}{mL^2}},\quad \mathbf{a}_2^{(3)} = (mL)^{-1/2}\begin{bmatrix} -0.236352 \\ 1.448321 \\ -0.040348 \end{bmatrix}$$

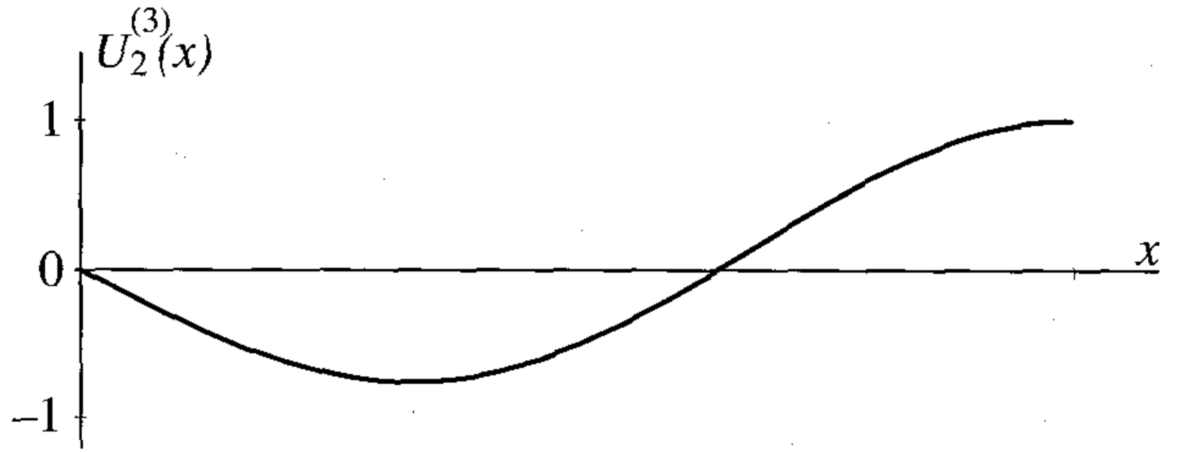
$$\omega_3^{(3)} = 7.931607\sqrt{\frac{EA}{mL^2}},\quad \mathbf{a}_3^{(3)} = (mL)^{-1/2}\begin{bmatrix} 0.097373 \\ -0.163450 \\ 1.432793 \end{bmatrix}$$

Example : ***n* = 3**

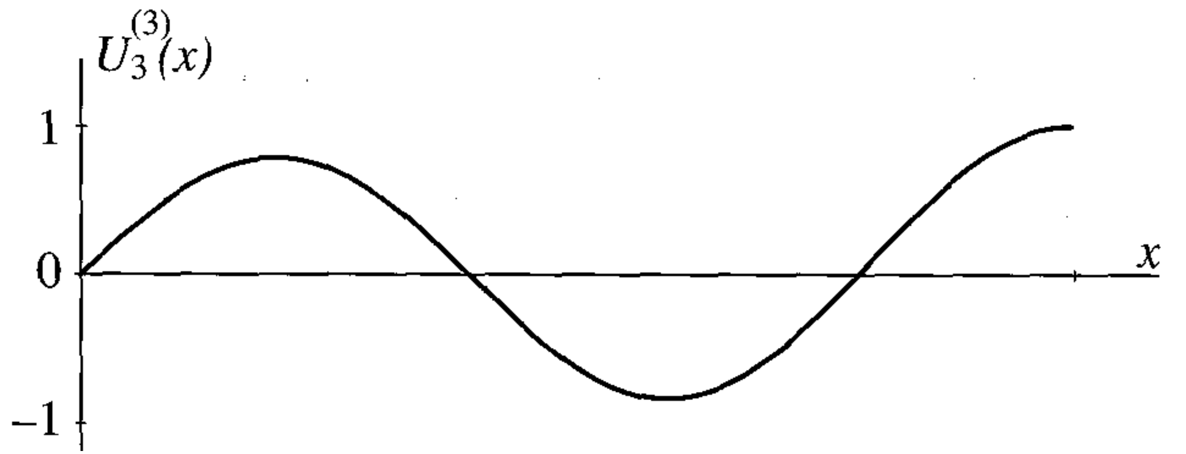
$$\omega_1 = 1.7742\sqrt{\frac{EA}{mL^2}}$$



$$\omega_2 = 4.8222\sqrt{\frac{EA}{mL^2}}$$



$$\omega_3 = 7.9316\sqrt{\frac{EA}{mL^2}}$$



Example :

The Ritz eigenvalues for the two approximations are:

$$\lambda_1^{(2)} = 3.148183EA/mL^2, \ \lambda_2^{(2)} = 23.284913EA/mL^2$$

$$\lambda_1^{(3)} = 3.147951EA/mL^2, \ \lambda_2^{(3)} = 23.253490EA/mL^2, \ \lambda_3^{(3)} = 62.910394EA/mL^2$$

- The improvement in the first two Ritz natural frequencies is very small,
  - indicates the chosen comparison functions resemble very closely the actual natural modes.
- Convergence to the lowest eigenvalue with six decimal places accuracy is obtained with 11 terms: $\lambda_1^{(11)} = 3.147888EA/mL^2$

Truncation

Approximation of a system with an infinite number of DOFs by a discrete system with n degrees of freedom implies truncation:

$$a_{n+1} = a_{n+2} = \ldots = 0$$

Constraints tend to increase the stiffness of a system:

$$\lambda_r^{(n)} \geq \lambda_r\,,\ r = 1, 2, \ldots, n$$

The nature of the Ritz eigenvalues requires further elaboration.

Truncation

A question of particular interest is how the eigenvalues $\lambda_r^{(n+1)}$ $(r = 1, 2, \ldots, n+1)$ of the (n +1)-DOF approximation relate to the eigenvalues $\lambda_r^{(n)}$ $(r = 1, 2, \ldots, n)$ of the n-DOF approximation.

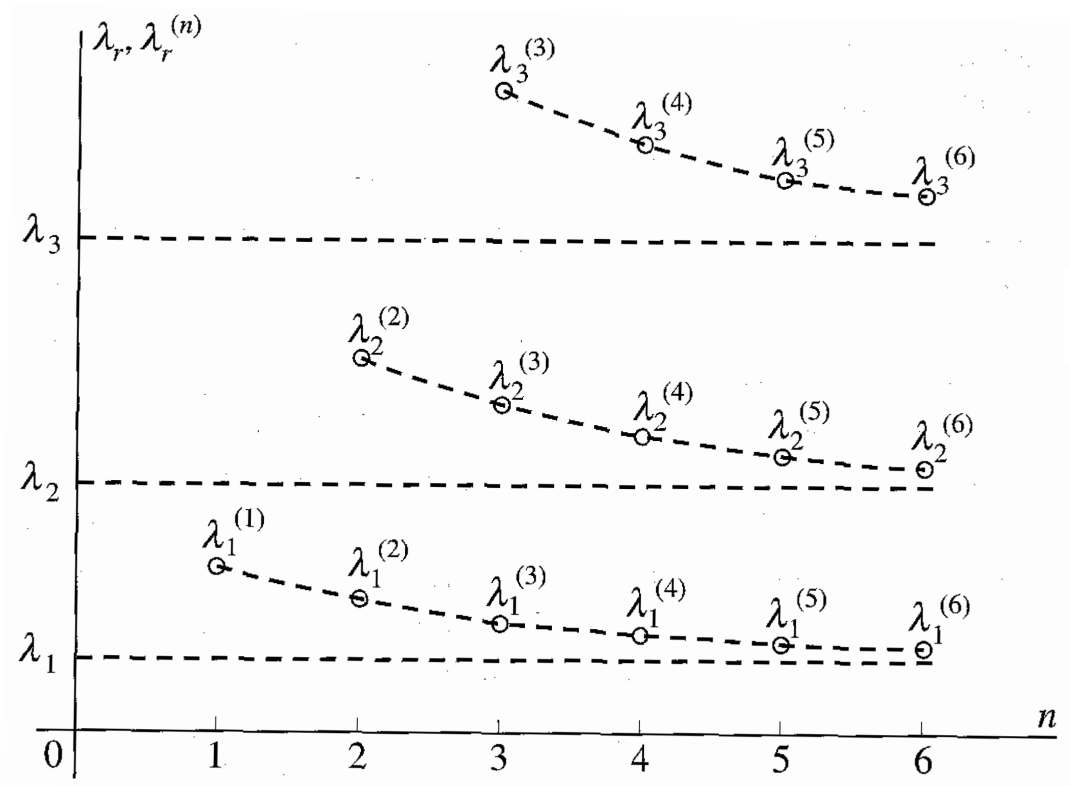
We observe that the extra term in series does not affect the mass and stiffness coefficients computed on the basis of an n-term series (embedding property):

$$M^{(n+1)} = \begin{bmatrix} & M^{(n)} & & x \\ & & & x \\ x & & x & x \end{bmatrix}, \quad K^{(n+1)} = \begin{bmatrix} & K^{(n)} & & x \\ & & & x \\ x & & x & x \end{bmatrix}$$

Truncation

For matrices with embedding property the eigenvalues satisfy the *separation theorem:*

$$\lambda_1^{(n+1)} \le \lambda_1^{(n)} \le \lambda_2^{(n+1)} \le \lambda_2^{(n)} \le \ldots \le \lambda_n^{(n+1)} \le \lambda_n^{(n)} \le \lambda_{n+1}^{(n+1)}$$



$$\lambda_r^{(n+1)} \le \lambda_r^{(n)}, \; r = 1, 2, \ldots, n$$

$$\lim_{n \to \infty} \lambda_r^{(n)} = \lambda_r, \; r = 1, 2, \ldots, n$$

Free Vibration of Continuous Systems

- Free Axial and Torsional Vibration
- Free Transverse Vibration of Bernoulli-Euler Beams
- Rayleigh's Method for Approximating the Fundamental Frequency of a Continuous System
- Free Transverse Vibration of Beams Including Shear Deformation and Rotatory Inertia
- Some Properties of Natural Modes of Continuous Systems
- Free Vibration of Thin Flat Plates

Structural Dynamics

Lecture 18: Free Vibration of Continuous Systems (Chapter 13)

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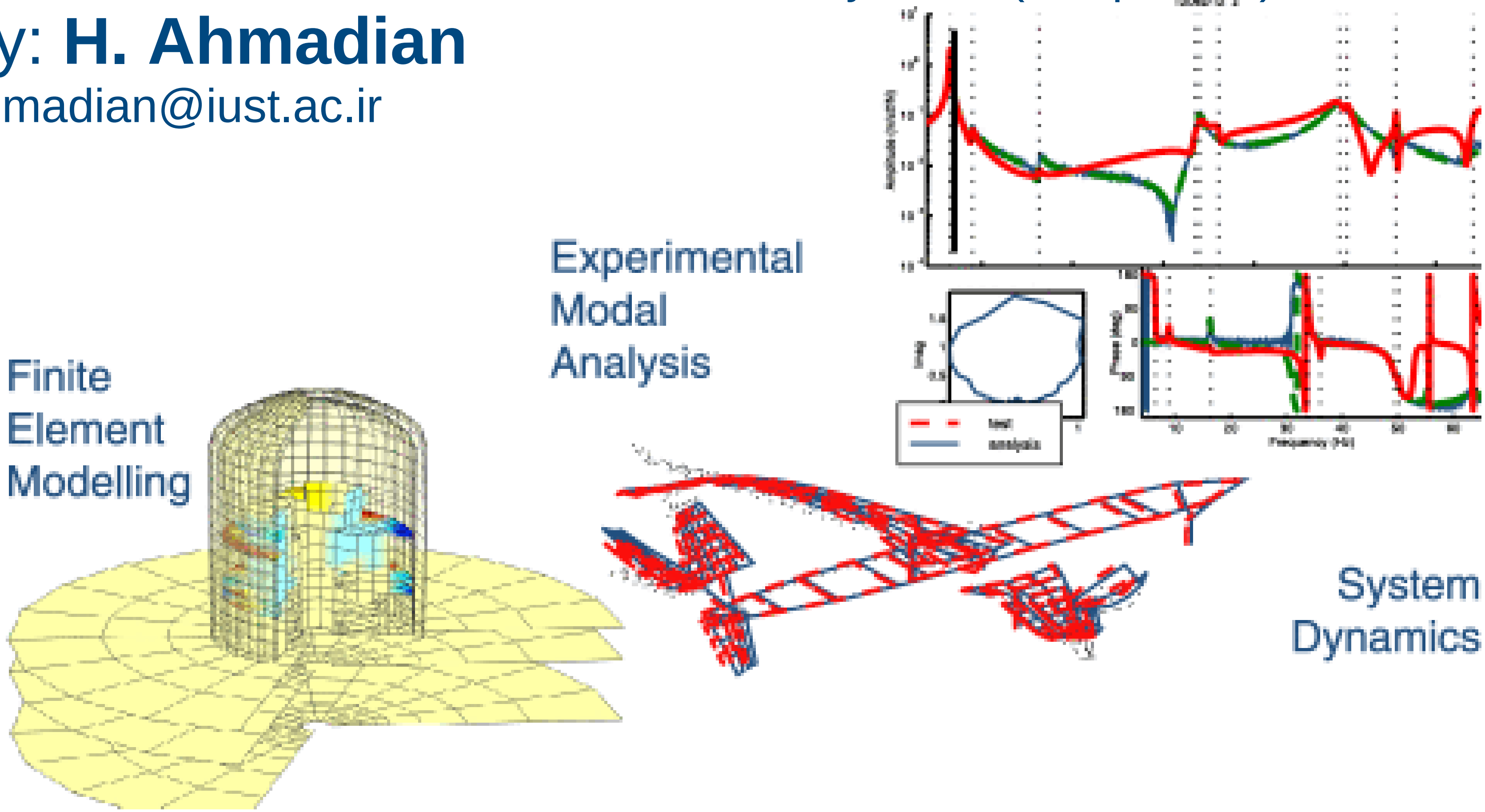

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Free Vibration of Continuous Systems

- > Free Axial and Torsional Vibration
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VIBRATION OF PLATES

- Plates have bending stiffness in a manner similar to beams in bending.
- In the case of plates one can think of two planes of bending, producing in general two distinct curvatures.
- The small deflection theory of thin plates, called *classical plate theory* or *Kirchhoff theory,* is based on assumptions similar to those used in thin beam or Euler-Bernoulli beam theory.

EQUATION OF MOTION: CLASSICAL PLATE THEORY

The *elementary theory of plates* is based on the following assumptions:

- The thickness of the plate (h) is small compared to its lateral dimensions.
- The middle plane of the plate does not undergo in-plane deformation. Thus, the midplane remains as the neutral plane after deformation or bending.
- The displacement components of the midsurface of the plate are small compared to the thickness of the plate.
- The influence of transverse shear deformation is neglected. This implies that plane sections normal to the midsurface before deformation remain normal to the rnidsurface even after deformation or bending.
- The transverse normal strain under transverse loading can be neglected. The transverse normal stress is small and hence can be neglected compared to the other components of stress.

Moment - Shear Force Resultants:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

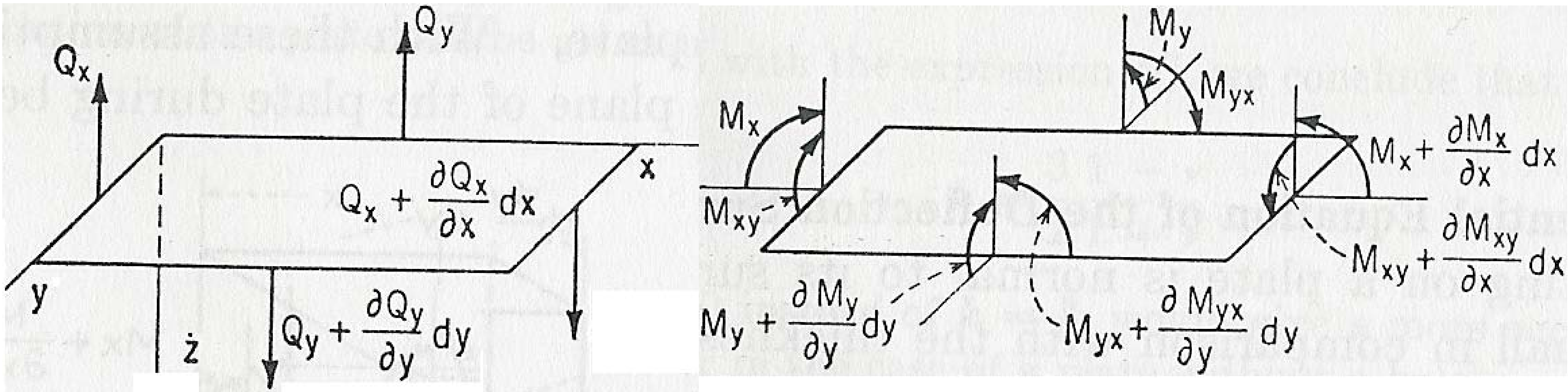
$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$

$$M_{xy} = M_{yx} = -(1-\nu)D\frac{\partial^2 w}{\partial x\,\partial y}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

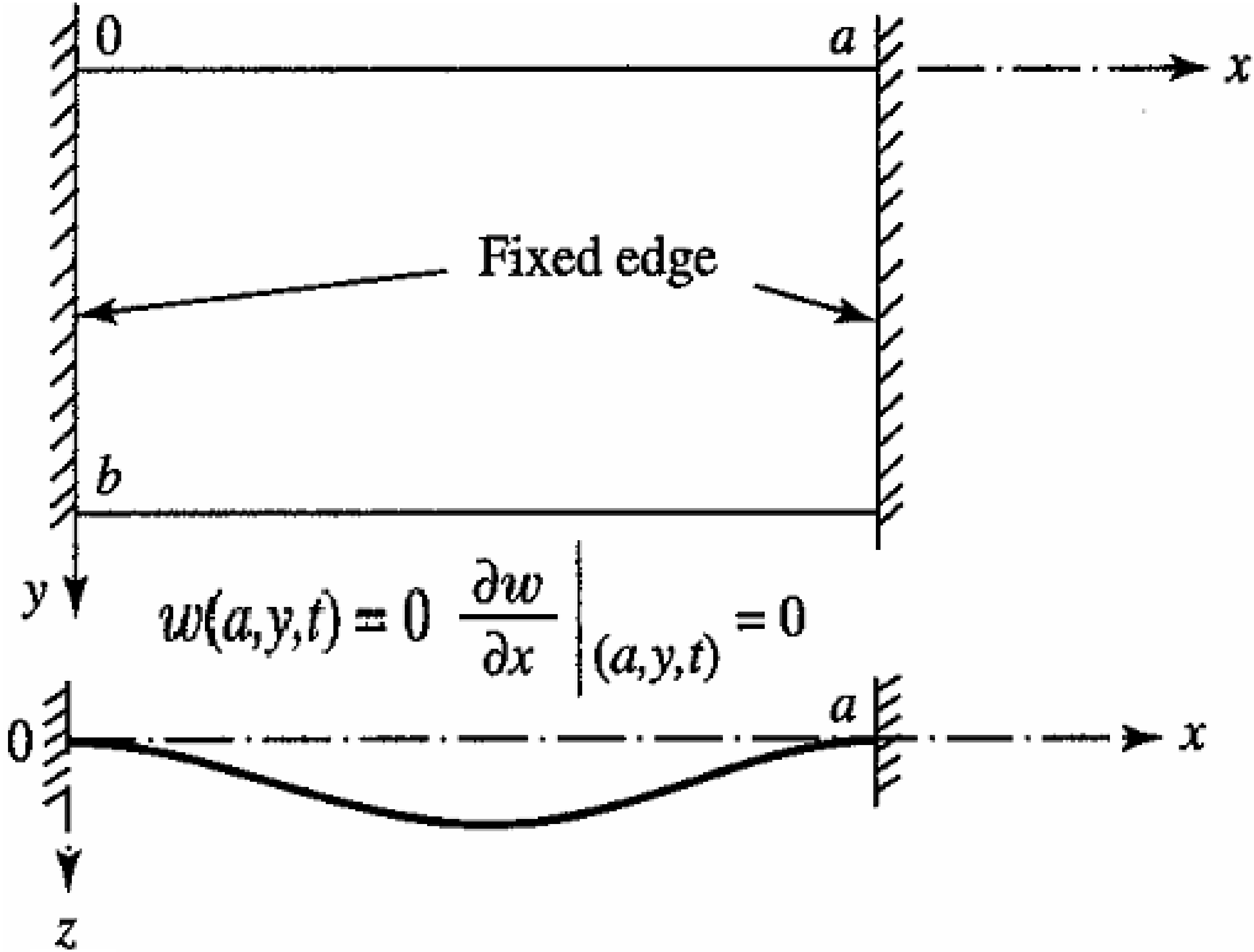
Equation of motion



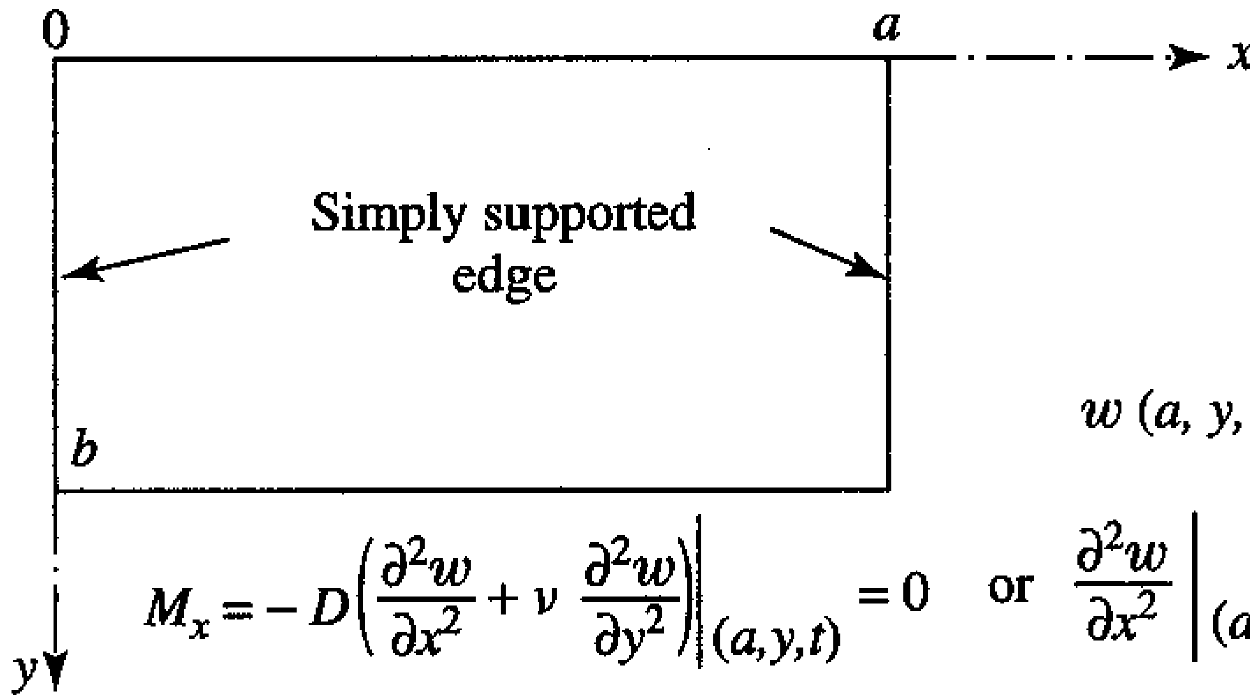
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + f(x,y,t) = \rho h \frac{\partial^2 w}{\partial t^2}$$

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho h \frac{\partial^2 w}{\partial t^2} = f(x,y,t)$$

BOUNDARY CONDITIONS

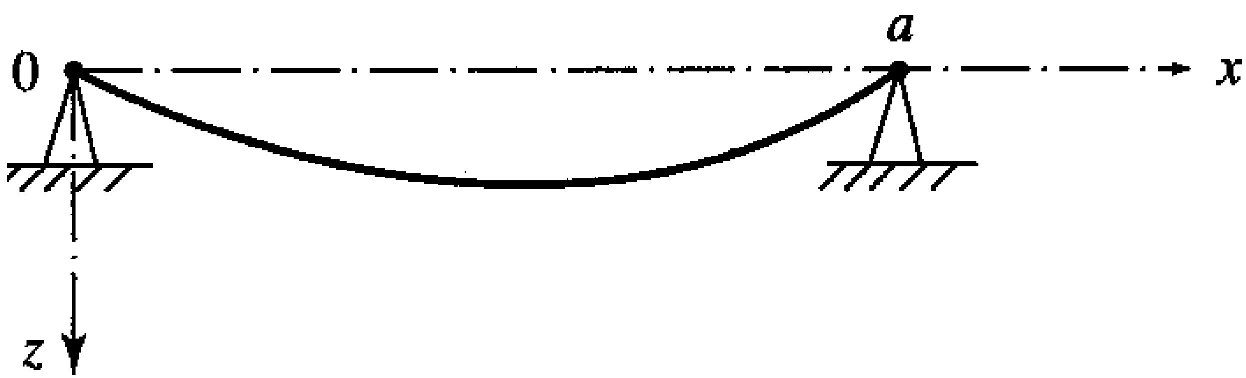


BOUNDARY CONDITIONS



$$w\,(a,\,y,\,t) = 0$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{(a,y,t)} = 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial x^2}\Bigg|_{(a,y,t)} = 0$$



BOUNDARY CONDITIONS: Free Edge

> There are three boundary conditions, whereas the equation of motion requires only two:

$$M_x|_{x=a} = 0 \quad Q_x|_{x=a} = 0 \quad M_{xy}|_{x=a} = 0$$

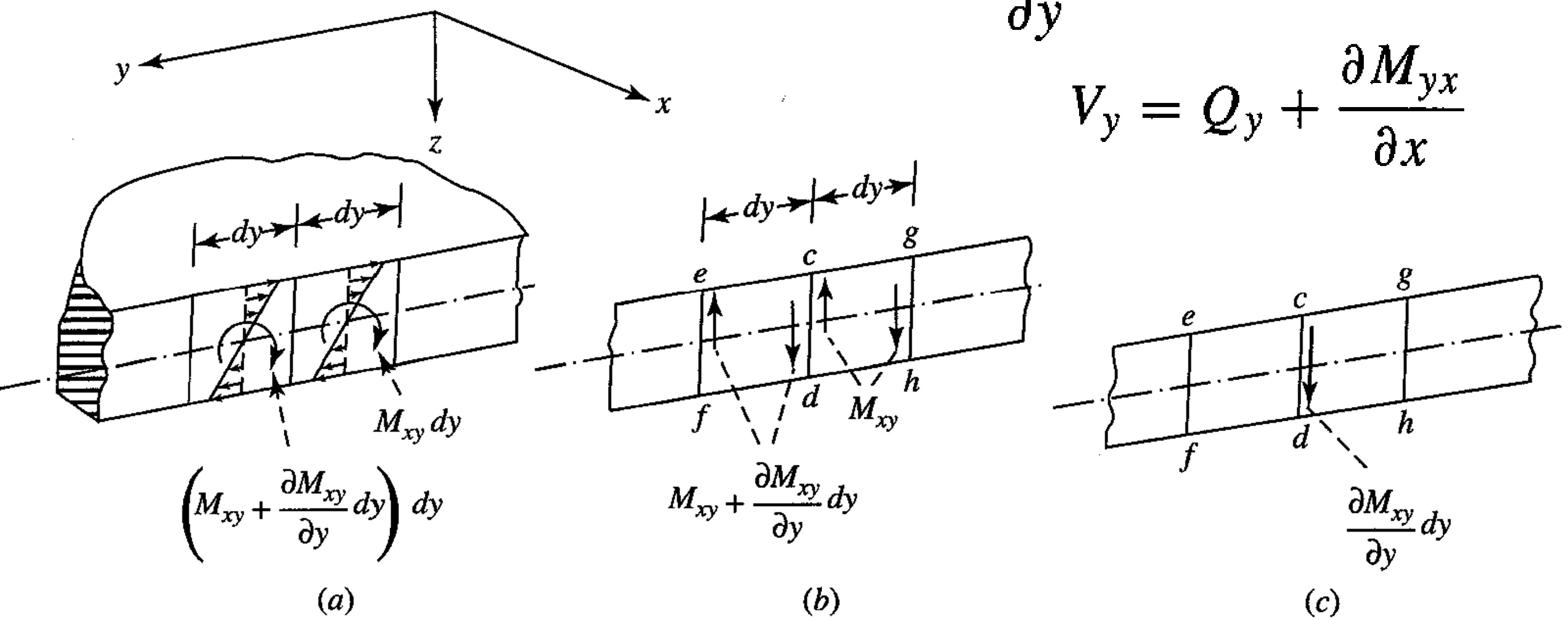
> Kirchhoff showed that the conditions on the shear force and the twisting moment are not independent and can be combined into only one boundary condition.

BOUNDARY CONDITIONS: Free Edge

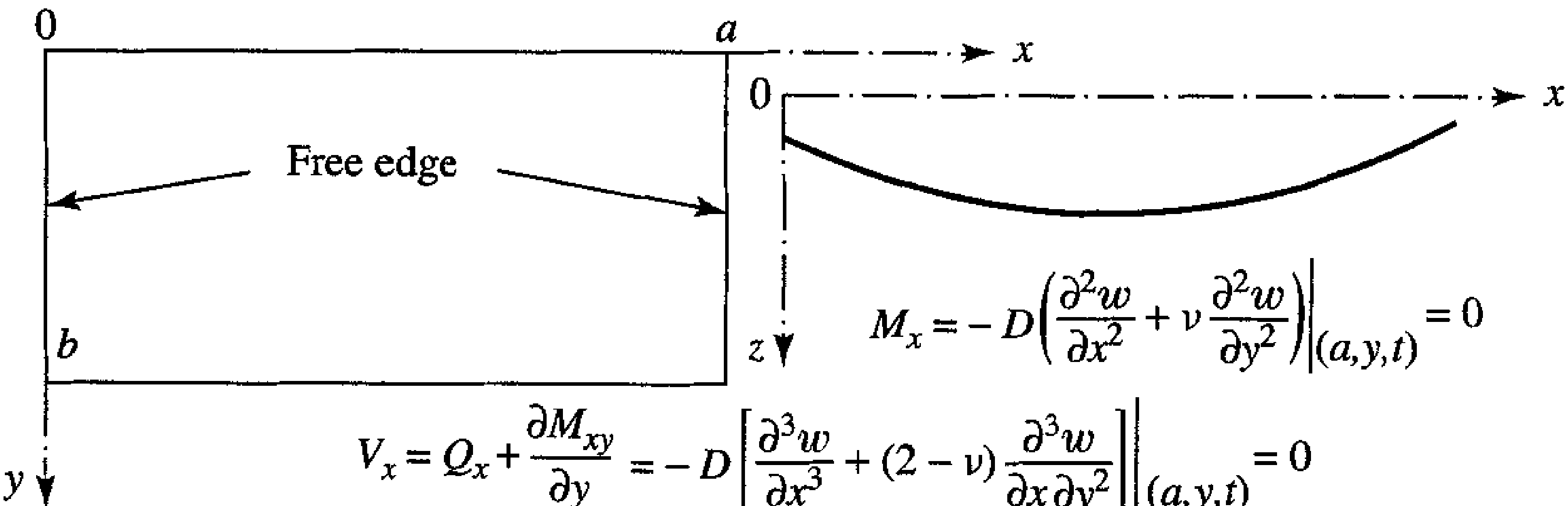
Replacing the twisting moment by an equivalent vertical force.

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}$$

$$V_y = Q_y + \frac{\partial M_{yx}}{\partial x}$$



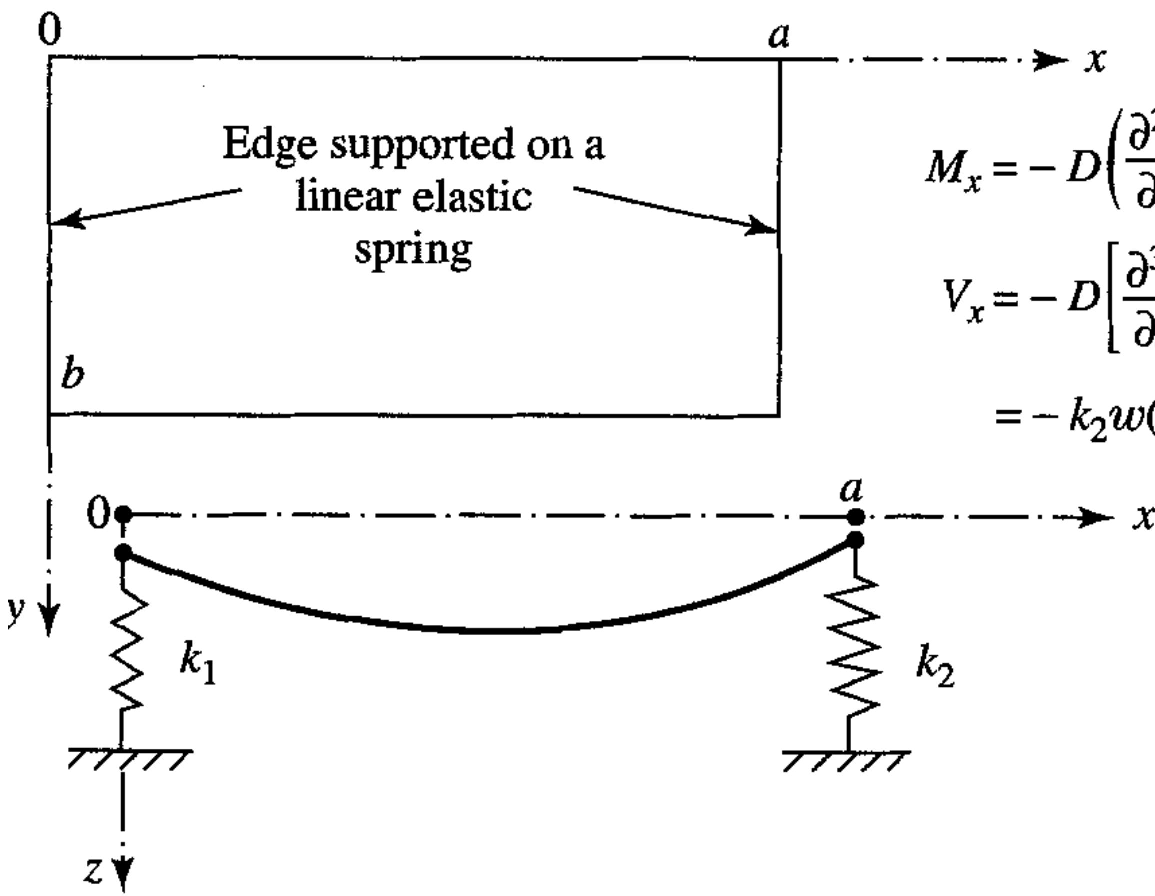
BOUNDARY CONDITIONS



$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)\bigg|_{(a,y,t)} = 0$$

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]\bigg|_{(a,y,t)} = 0$$

BOUNDARY CONDITIONS

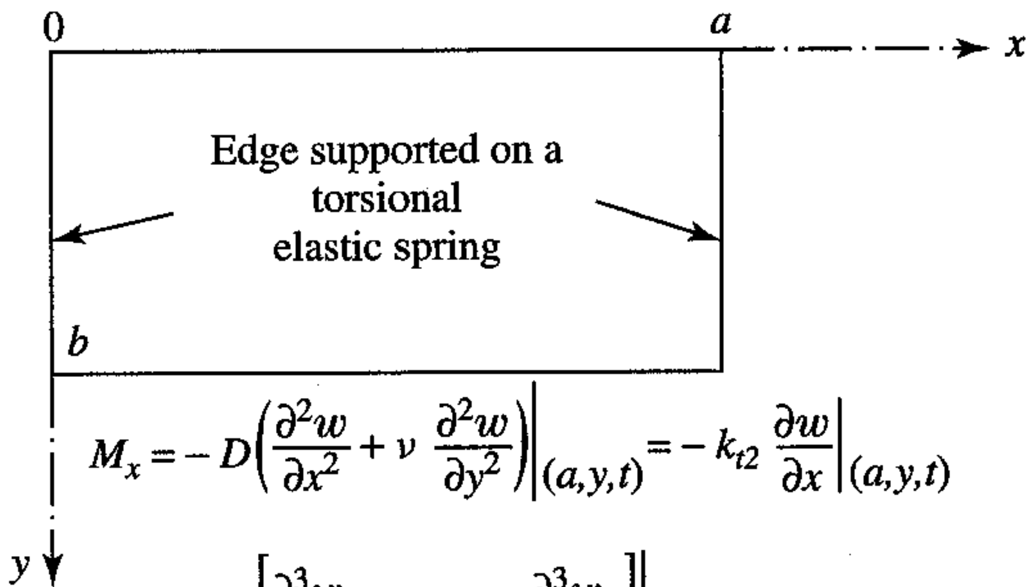


$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)\bigg|_{(a,y,t)} = 0$$

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\,\frac{\partial^3 w}{\partial x\,\partial y^2}\right]\bigg|_{(a,y,t)}$$

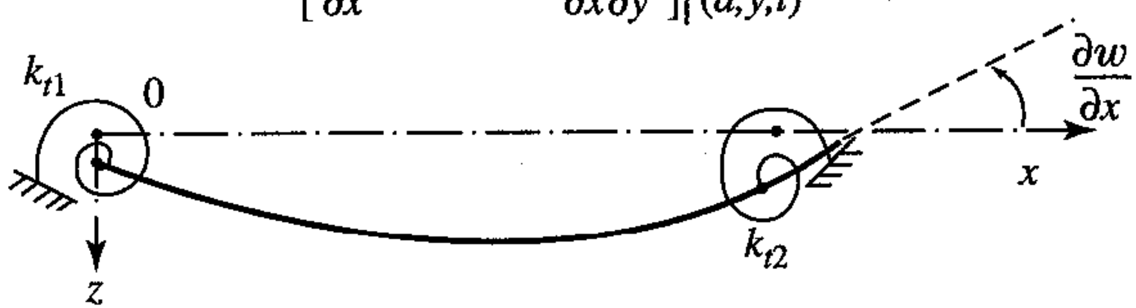
$$= -k_2 w(a,y,t)$$

BOUNDARY CONDITIONS



$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{(a,y,t)} = -k_{t2}\,\frac{\partial w}{\partial x}\bigg|_{(a,y,t)}$$

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\,\frac{\partial^3 w}{\partial x\,\partial y^2}\right]\Bigg|_{(a,y,t)} = 0$$



FREE VIBRATION OF RECTANGULAR PLATES

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=0$$

$$w(x,y,t)=W(x,y)T(t)$$

$$\frac{d^2T(t)}{dt^2}+\omega^2T(t)=0 \qquad T(t)=A\cos\omega t+B\sin\omega t$$

$$\nabla^4W(x,y)-\lambda^4W(x,y)=0 \qquad \lambda^4=\frac{\rho h\omega^2}{D}$$

FREE VIBRATION OF RECTANGULAR PLATES

$$(\nabla^4 - \lambda^4)W(x,y) = (\nabla^2 + \lambda^2)(\nabla^2 - \lambda^2)W(x,y) = 0$$

$$(\nabla^2 + \lambda^2)W_1(x,y) = \frac{\partial^2 W_1}{\partial x^2} + \frac{\partial^2 W_1}{\partial y^2} + \lambda^2 W_1(x,y) = 0$$

$$(\nabla^2 - \lambda^2)W_2(x,y) = \frac{\partial^2 W_2}{\partial x^2} + \frac{\partial^2 W_2}{\partial y^2} - \lambda^2 W_2(x,y) = 0$$

FREE VIBRATION OF RECTANGULAR PLATES

$$
\begin{aligned}
W(x, y) = {} & A_1 \sin \alpha x \sin \beta y + A_2 \sin \alpha x \cos \beta y \\
& + A_3 \cos \alpha x \sin \beta y + A_4 \cos \alpha x \cos \beta y \\
& + A_5 \sinh \theta x \sinh \phi y + A_6 \sinh \theta x \cosh \phi y \\
& + A_7 \cosh \theta x \sinh \phi y + A_8 \cosh \theta x \cosh \phi y
\end{aligned}
$$

$$\lambda^2 = \alpha^2 + \beta^2 = \theta^2 + \phi^2$$

Solution for a Simply Supported Plate

$$W(0,y)=\frac{d^2W}{dx^2}(0,y)=W(a,y)=\frac{d^2W}{dx^2}(a,y)=0$$

$$W(x,0)=\frac{d^2W}{dy^2}(x,0)=W(x,b)=\frac{d^2W}{dy^2}(x,b)=0$$

We find that all the constants A_i except A_1 *and*

$$\sin\alpha a=0 \longrightarrow \alpha_m a=m\pi, \qquad m=1,2,\ldots$$

$$\sin\beta b=0 \longrightarrow \beta_n b=n\pi, \qquad n=1,2,\ldots$$

$$\omega_{mn}=\lambda_{mn}^2\left(\frac{D}{\rho h}\right)^{1/2}=\pi^2\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]\left(\frac{D}{\rho h}\right)^{1/2},$$

$$W_{mn}(x,y)=A_{1mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad m,n=1,2,\ldots$$

Solution for a Simply Supported Plate

$$w_{mn}(x,y,t) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t)$$

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t)$$

The initial conditions of the plate are:

$$w(x,y,0) = w_0(x,y)$$

$$\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$$

Solution for a Simply Supported Plate

$$w(x,y,0) = w_0(x,y)$$

$$\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$$

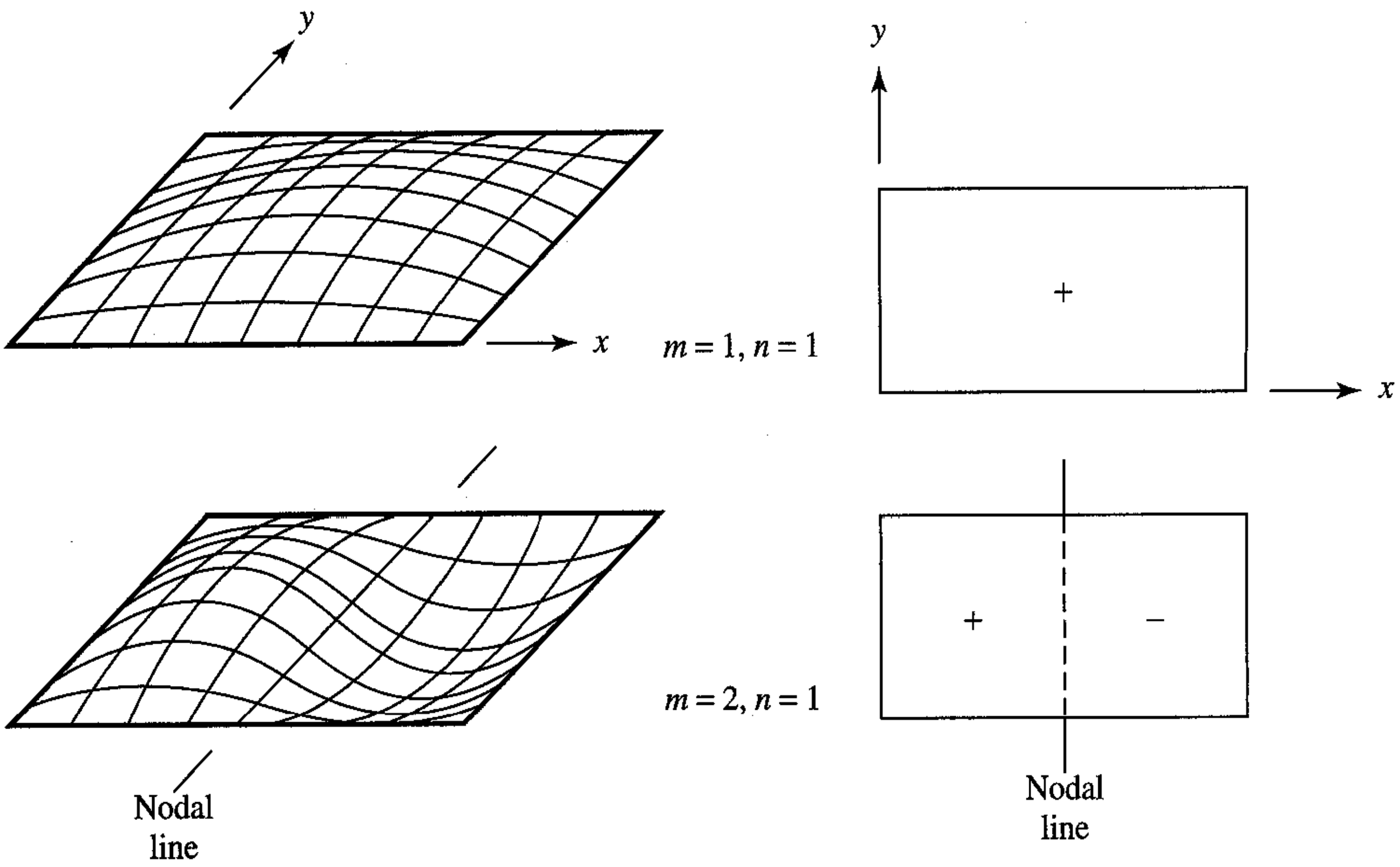
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = w_0(x,y)$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\omega_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = \dot{w}_0(x,y)$$

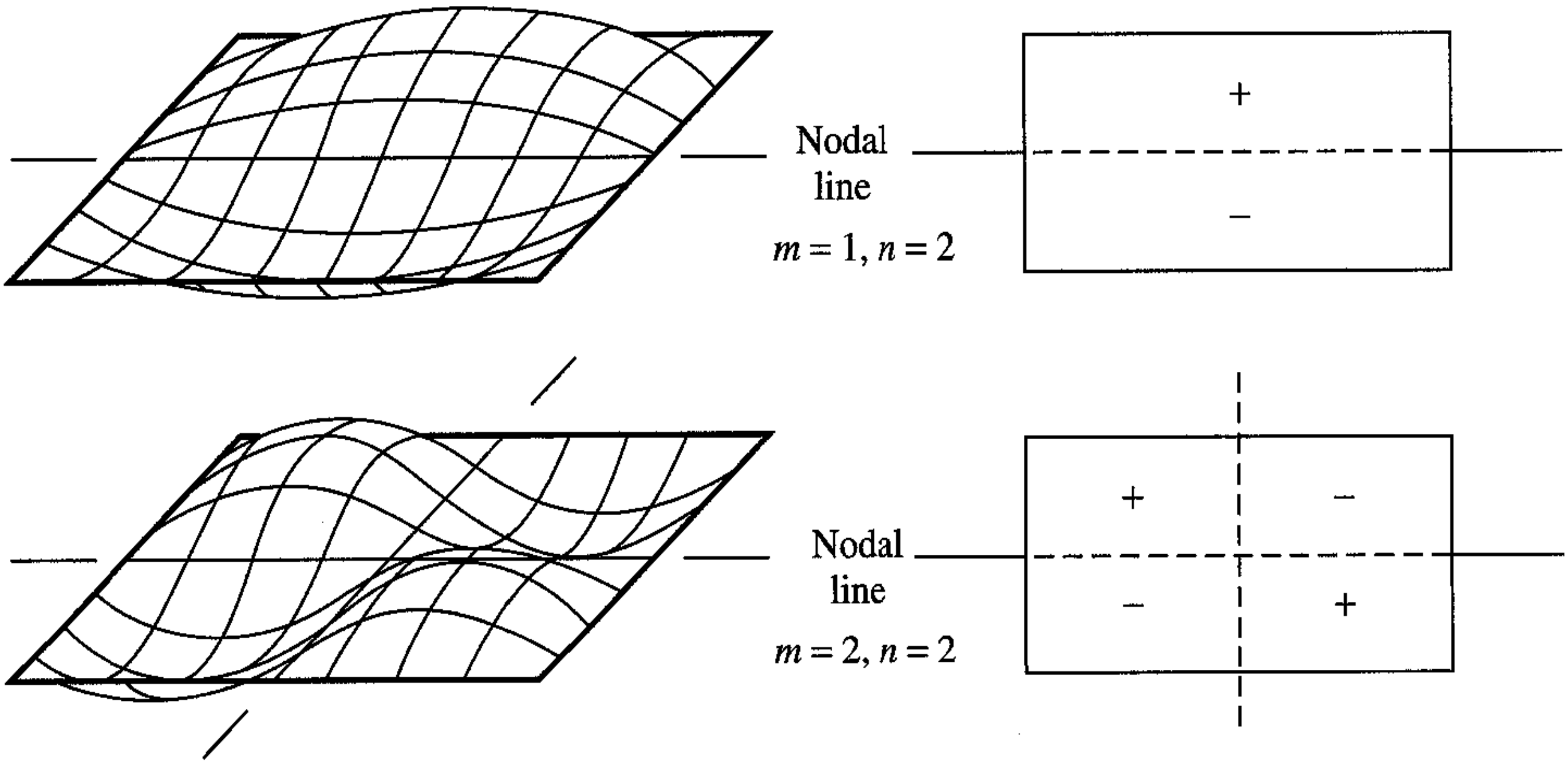
$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b w_0(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

$$B_{mn} = \frac{4}{ab\omega_{mn}}\int_0^a\int_0^b \dot{w}_0(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

Solution for a Simply Supported Plate



Solution for a Simply Supported Plate



Free Vibration of Continuous Systems

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Structural Dynamics

Lecture 19: Free Vibration of Continuous Systems (Chapter 13)

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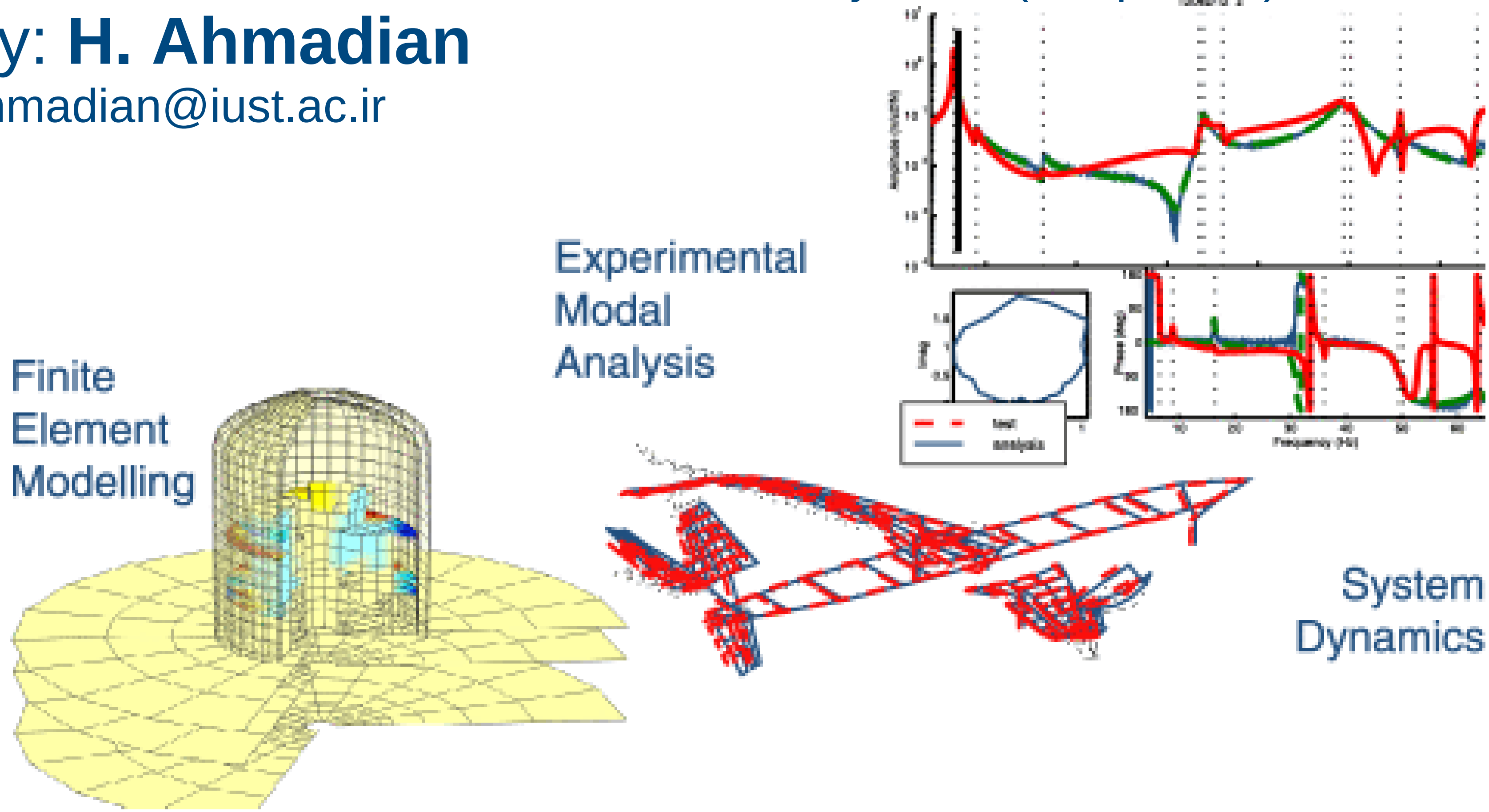

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Free Vibration of Continuous Systems

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> Free Vibration of Thin Flat Plates

Vibrations of Rectangular Plates

$$\nabla^4 W(x, y) - \lambda^4 W(x, y) = 0 \quad \lambda^4 = \frac{\rho h \omega^2}{D}$$

$$W(x,y) = X(x)Y(y)$$

$$X''''Y + 2X''Y'' + XY'''' - \lambda^4 XY = 0$$

The functions *X(x)* and *Y(y)* can be separated provided either of the followings are satisfied:

$$\Longrightarrow Y''(y) = -\beta^2 Y(y), Y''''(y) = -\beta^2 Y''(y)$$

$$\Longrightarrow X''(x) = -\alpha^2 X(x), X''''(x) = -\alpha^2 X''(x)$$

Vibrations of Rectangular Plates

$$Y''(y) = -\beta^2 Y(y), Y''''(y) = -\beta^2 Y''(y)$$
$$X''(x) = -\alpha^2 X(x), X''''(x) = -\alpha^2 X''(x)$$

These equations can be satisfied only by the trigonometric functions:

$$\begin{Bmatrix} \sin \alpha_m x \\ \cos \alpha_m x \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \sin \beta_n y \\ \cos \beta_n y \end{Bmatrix}$$

$$\alpha_m = \frac{m\pi}{a}, m = 1, 2, \ldots, \beta_n = \frac{n\pi}{b}, n = 1, 2, \ldots$$

Vibrations of Rectangular Plates

Assume that the plate is simply supported along edges *x =0* and *x =a*:

$$X_m(x) = A \sin \alpha_m x, \qquad m = 1, 2, \ldots$$

$$X_m(0) = X_m(a) = X''_m(0) = X''_m(a) = 0$$

Implying:

$$w(0,y,t) = w(a,y,t) = \nabla^2 w(0,y,t) = \nabla^2 w(a,y,t) = 0$$

$$Y''''(y) - 2\alpha_m^2 Y''(y) - (\lambda^4 - \alpha_m^4) Y(y) = 0$$

Vibrations of Rectangular Plates

The various boundary conditions can be stated,

SS-SS-SS-SS, SS-C-SS-C, SS-F-SS-F, SS-C-SS-SS, SS-F-SS-SS, SS-F-SS-C

Assuming: $\lambda^4 > \alpha_m^4$

$$Y(y) = e^{sy}$$

$$s^4 - 2s^2\alpha_m^2 - (\lambda^4 - \alpha_m^4) = 0$$

$$s_{1,2} = \pm\sqrt{\lambda^2 + \alpha_m^2}, \qquad s_{3,4} = \pm i\sqrt{\lambda^2 - \alpha_m^2}$$

$$Y(y) = C_1 \sin \delta_1 y + C_2 \cos \delta_1 y + C_3 \sinh \delta_2 y + C_4 \cosh \delta_2 y$$

$$\delta_1 = \sqrt{\lambda^2 - \alpha_m^2}, \qquad \delta_2 = \sqrt{\lambda^2 + \alpha_m^2}$$

Vibrations of Rectangular Plates

y = 0 and y = b are simply supported:

$$W(x,0)=0 \qquad\qquad Y(0)=0$$

$$W(x,b)=0 \qquad\qquad Y(b)=0$$

$$M_y(x,0)=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)\bigg|_{(x,0)}=0 \quad\Longrightarrow\quad \frac{d^2Y(0)}{dy^2}=0$$

$$M_y(x,b)=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right)\bigg|_{(x,b)}=0 \qquad\qquad \frac{d^2Y(b)}{dy^2}=0$$

$$C_2+C_4=0$$

$$C_1\sin\delta_1 b+C_2\cos\delta_1 b+C_3\sinh\delta_2 b+C_4\cosh\delta_2 b=0$$

$$-\delta_1^2 C_2+\delta_2^2 C_4=0$$

$$-C_1\delta_1^2\sin\delta_1 b-C_2\delta_1^2\cos\delta_1 b+C_3\delta_2^2\sinh\delta_2 b+C_4\delta_2^2\cosh\delta_2 b=0$$

$$\Longrightarrow\quad C_4=0,\quad C_2=0,\quad C_3=0$$

Vibrations of Rectangular Plates

y = 0 and y = b are simply supported:

$$\sin \delta_1 b = 0 \qquad \delta_1 = \frac{n\pi}{b}, \qquad n = 1,\ 2,\ \ldots.$$

$$Y_n(y) = C_1 \sin \delta_1 y = C_1 \sin \frac{n\pi y}{b}$$

$$\delta_1 = \sqrt{\lambda^2 - \alpha_m^2}, \Longrightarrow \lambda_{mn}^2 = \alpha_m^2 + \beta_n^2, \qquad \beta_n = \frac{n\pi}{b}$$

$$\omega_{mn} = \lambda_{mn}^2 \sqrt{\frac{D}{\rho h}} = (\alpha_m^2 + \beta_n^2)\sqrt{\frac{D}{\rho h}} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}}, \qquad m, n = 1,\ 2, \ldots$$

$$W_{mn}(x, y) = C_{mn} \sin \alpha_m x \ \sin \beta_n y, \qquad m, n = 1,\ 2, \ldots$$

Vibrations of Rectangular Plates

y = 0 and y = b are clamped:

$$Y(0)=0 \qquad \frac{dY}{dy}(0)=0 \qquad Y(b)=0 \qquad \frac{dY}{dy}(b)=0$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \delta_1 & 0 & \delta_2 & 0 \\ \sin\delta_1 b & \cos\delta_1 b & \sinh\delta_2 b & \cosh\delta_2 b \\ \delta_1\cos\delta_1 b & -\delta_1\sin\delta_1 b & \delta_2\cosh\delta_2 b & \delta_2\sinh\delta_2 b \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$2\delta_1\delta_2(\cos\delta_1 b\ \cosh\delta_2 b - 1) - \alpha_m^2 \sin\delta_1 b\ \sinh\delta_2 b = 0$$

$$Y_n(y) = C_n[(\cosh\delta_2 b - \cos\delta_1 b)\ (\delta_1\sinh\delta_2 y - \delta_2\sin\delta_1 y)$$
$$- (\delta_1\sinh\delta_2 b - \delta_2\sin\delta_1 b)\ (\cosh\delta_2 y - \cos\delta_1 y)]$$

$$W_{mn}(x,y) = C_{mn} Y_n(y)\ \sin\alpha_m x$$

Vibrations of Rectangular Plates

Table 14.1 Frequency Equations and Mode Shapes of Rectangular Plates with Different Boundary Conditions[a]

Case	Boundary conditions	Frequency equation	y-mode shape, $Y_n(y)$ without a multiplication factor, where $W_{mn}(x,y) = C_{mn} X_m(x)\, Y_n(y)$, with $X_m(x) = \sin\alpha_m x$
1	SS-SS-SS-SS	$\sin\delta_1 b = 0$	$Y_n(y) = \sin\beta_n y$
2	SS-C-SS-C	$2\delta_1\delta_2(\cos\delta_1 b\ \cosh\delta_2 b - 1) - \alpha_m^2 \sin\delta_1 b\ \sinh\delta_2 b = 0$	$Y_n(y) = (\cosh\delta_2 b - \cos\delta_1 b)\,(\delta_1 \sinh\delta_2 y - \delta_2 \sin\delta_1 y)$ $-(\delta_1 \sinh\delta_2 b - \delta_2 \sin\delta_1 b)\,(\cosh\delta_2 y - \cos\delta_1 y)$
3	SS-F-SS-F	$\sinh\delta_2 b\ \sin\delta_1 b\ \{\delta_2^2[\lambda^2 - \alpha_m^2(1-\nu)]^4$ $-\delta_1^2[\lambda^2 + \alpha_m^2(1-\nu)]^4\}$ $-2\delta_1\delta_2[\lambda^4 - \alpha_m^4(1-\nu)^2]^2\ (\cosh\ \delta_2 b\ \cos\delta_1 b - 1) = 0$	$Y_n(y) = -(\cosh\delta_2 b - \cos\delta_1 b)\ [\lambda^4 - \alpha_m^4(1-\nu)^2]$ $\{\delta_1\ [\lambda^2 + \alpha_m^2(1-\nu)]\ \sinh\delta_2 y$ $+\delta_2\ [\lambda^2 - \alpha_m^2(1-\nu)]\ \sin\delta_1 y\} + \{\delta_1\ [\lambda^2 + \alpha_m^2(1-\nu)]^2\ \sinh\delta_2 b$ $-\delta_2\ [\lambda^2 - \alpha_m^2(1-\nu)]^2\ \sin\delta_1 b\}\{[\lambda^2 - \alpha_m^2(1-\nu)]\ \cosh\ \delta_2 y$ $+[\lambda^2 + \alpha_m^2(1-\nu)]\ \cos\delta_1 y\}$
4	SS-C-SS-SS	$\delta_2 \cosh\delta_2 b\ \sin\delta_1 b - \delta_1 \sinh\delta_2 b\ \cos\delta_1 b = 0$	$Y_n(y) = \sin\delta_1 b\ \sinh\delta_2 y - \sinh\delta_2 b\ \sin\delta_1 y$
5	SS-F-SS-SS	$\delta_2[\lambda^2 - \alpha_m^2(1-\nu)]^2\ \cosh\delta_2 b\ \sin\delta_1 b$ $-\delta_1[\lambda^2 + \alpha_m^2(1-\nu)]^2 \sinh\delta_2 b\ \cos\delta_1 b = 0$	$Y_n(y) = [\lambda^2 - \alpha_m^2(1-\nu)]\ \sin\delta_1 b\ \sinh\delta_2 y$ $+[\lambda^2 + \alpha_m^2(1-\nu)]\ \sinh\delta_2 b\ \sin\delta_1 y$
6	SS-F-SS-C	$\delta_1\delta_2\ [\lambda^4 - \alpha_m^4(1-\nu)^2] + \delta_1\delta_2[\lambda^4 + \alpha_m^4\ (1-\nu)^2]$ $\cdot \cosh\delta_2 b\ \cos\delta_1 b + \alpha_m^2[\lambda^4(1-2\nu) - \alpha_m^4(1-\nu)^2]$ $\cdot \sinh\delta_2 b\ \sin\delta_1 b = 0$	$Y_n(y) = \{[\lambda^2 + \alpha_m^2(1-\nu)]\ \cosh\delta_2 b + [\lambda^2 - \alpha_m^2(1-\nu)]\ \cos\delta_2 b\}$ $\cdot(\delta_2 \sin\delta_1 y - \delta_1 \sinh\delta_2 y) + \{\delta_1[\lambda^2 + \alpha_m^2(1-\nu)]\ \sinh\ \delta_2 b$ $+\delta_2[\lambda^2 - \alpha_m^2\ (1-\nu)]\ \sin\delta_1 b\}\ (\cosh\delta_2 y - \cos\delta_1 y)$

Source: Refs. [1] and [2].

[a] Edges $x = 0$ and $x = a$ simply supported.

Vibrations of Rectangular Plates

Exact characteristic equations for some of classical boundary conditions of vibrating moderately thick rectangular plates

Shahrokh Hosseini Hashemi and M. Arsanjani ,International Journal of Solids and Structures Volume 42, Issues 3-4, February 2005, Pages 819-853

Exact solution for linear buckling of rectangular Mindlin plates

Shahrokh Hosseini-Hashemi, Korosh Khorshidi, and Marco Amabili, Journal of Sound and Vibration Volume 315, Issues 1-2, 5 August 2008, Pages 318-342

FORCED VIBRATION OF RECTANGULAR PLATES

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}(x,y)\eta_{mn}(t)$$

the normal modes

$$\int_0^a \int_0^b \rho h W_{mn}^2 \, dx \, dy = 1$$

FORCED VIBRATION OF RECTANGULAR PLATES

Using a modal analysis procedure:

$$\ddot{\eta}_{mn}(t) + \omega_{mn}^2 \eta_{mn}(t) = N_{mn}(t), \qquad m, n = 1, 2, \ldots$$

$$N_{mn}(t) = \int_0^a \int_0^b W_{mn}(x,y) f(x,y,t)\, dx\, dy$$

$$\eta_{mn}(t) = \eta_{mn}(0) \cos \omega_{mn} t + \frac{\dot{\eta}_{mn}(0)}{\omega_{mn}} \sin \omega_{mn} t + \frac{1}{\omega_{mn}} \int_0^t N_{mn}(\tau) \sin \omega_{mn}(t - \tau)\, d\tau$$

FORCED VIBRATION OF RECTANGULAR PLATES

The response of simply supported rectangular plates:

$$W_{mn}(x,y) = A_{1mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \qquad m,n = 1,2,\ldots$$

$$A_{1mn} = 2/\sqrt{\rho h a b}$$

$$\omega_{mn} = \pi^2 \left(\frac{D}{\rho h}\right)^{1/2} \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$$

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \eta_{mn}(0) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos\left[\pi^2 \sqrt{\frac{D}{\rho h}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) t\right]$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\dot{\eta}_{mn}(0)(\rho h)^{1/2}}{\pi^2 (D)^{1/2}} \frac{1}{m^2/a^2 + n^2/b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\sin\left[\pi^2 \sqrt{\frac{D}{\rho h}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) t\right]$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(\rho h)^{1/2}}{\pi^2 D^{1/2}} \frac{1}{m^2/a^2 + n^2/b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \int_0^t N_{mn}(\tau)$$

$$\sin\left[\pi^2 \sqrt{\frac{D}{\rho h}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) (t-\tau)\right] d\tau$$

Free Vibration of Continuous Systems

> Free Axial and Torsional Vibration

> Free Transverse Vibration of Bernoulli-Euler Beams

> Rayleigh's Method for Approximating the Fundamental Frequency of a Continuous System

> Free Transverse Vibration of Beams Including Shear Deformation and Rotatory Inertia

> Some Properties of Natural Modes of Continuous Systems

> Free Vibration of Thin Flat Plates