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FINITE ELEMENT MODEL IDENTIFICATION USING MODAL DATA

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This paper describes a procedure for reconstructing a finite element model of a structure from modal data and connectivity information. This is an example of a classical ill-posed problem; it is regularized by requiring that the model be near to a given one. The procedure is illustrated by constructing a finite element model of a cantilever beam from two or three modes and frequencies.

1. INTRODUCTION

This paper is concerned with modelling the dynamics, in particular the lightly damped infinitesimal vibration, of a structure. A common modelling situation is one in which the researcher has some results predicted from an analytical model on the one hand and some experimentally acquired modal and/or natural frequency data on the other, and wishes to bring them into agreement. How one brings them into agreement depends on the relative confidence one has in the analytical model and in the experimental results. If the model is trusted and the predicted values do not agree with the experimental results, then the latter must be modified: sometimes experimental mode shapes are modified so that they are orthogonal with respect to a given mass matrix. If, on the other hand, the experimental results are given primacy, then the model must be modified. We are concerned primarily with the latter situation: a recent bibliography is given by Denman and Husselman [1] and comments on the literature may be found in the papers cited below, particularly in those by Baruch and Bar Itzhack [2], Berman and Nagy [3] and Kabe [4]; we comment only on those papers which relate closely to our concerns.

When a structure is excited by impact with a hammer, or in sinusoidal vibration with a shaker, the test results are analyzed by using the equations governing damped free vibration; namely,

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0},$$

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where **M**, **C** and **K** are the mass, damping and stiffness matrices, all square matrices of order N. We are not interested, at this time, in identifying **C**; like others, we will interpret the test results by omitting **C** altogether, so that the free vibration of the structure is governed by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

and individual modes

$$\mathbf{q} = \boldsymbol{\phi}_r \sin \omega_r t$$

are governed by

$$\mathbf{K}\boldsymbol{\phi}_r = \omega_r^2 \mathbf{M}\boldsymbol{\phi}_r. \tag{1}$$

An immediate consequence of this equation is that modes corresponding to different frequencies ω_r and ω_s are orthogonal with respect to the mass matrix, and the stiffness matrix:

$$\boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_s = \delta_{rs}, \qquad \boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_s = \omega_r^2 \delta_{rs}. \tag{2, 3}$$

We are now in a position to start commenting on previous work. Baruch and Bar Itzhack [2] assumed that **M** was known precisely. They took the measured modes $\boldsymbol{\psi}_r$ and modified them to obtain modes $\boldsymbol{\phi}_r$, satisfying equation (2). They then used a Lagrange multiplier technique to find a stiffness matrix **K** satisfying equations (1), which was as near as possible, in norm, to a given stiffness matrix $\mathbf{K}_A$. This procedure provided a promising step in the solution, but had many deficiencies: it assumed that **M** was known; even if $\mathbf{K}_A$ had a definite form corresponding to a physical model there was no certainty that **K**, the stiffness matrix that was derived, would have this same form.

Baruch and Bar Itzhack modified the measured modes to conform to the orthogonality with respect to the known mass matrix; Berman [5], on the other hand, assumed that the measured modes were correct and found a mass matrix **M** as near as possible, in norm, to a given matrix $\mathbf{M}_A$, so that the measured modes were orthogonal to **M**. Berman and Nagy [3] combined this step with Baruch and Bar Itzhack's improvement of the stiffness matrix to give the analytical model improvement (AMI) method; see also Wei [6, 7]. The main deficiency in this method is that even if $\mathbf{M}_A$ and $\mathbf{K}_A$ have definite forms corresponding to some physical model, **M** and **K** will not necessarily retain these forms. Kabe [4, 8, 9] attempted to remove this deficiency by restricting the form of **K**, assuming that **M** was known. He supposed that **K** had the same pattern of zero and non-zero terms as did $\mathbf{K}_A$, so preserving the connectivity of the model; he then had to seek the factors γ_{ij} by which a term k_{ij} in **K** was obtained from the term $(k_A)_{ij}$ in $\mathbf{K}_A$. Kammer [10] formulated the same problem using the language of projections and the Moore Penrose inverse, so simplifying the analysis.

The roots of the method to be described in the present paper lie in the work of Berman and Flannelly [11], O'Callaghan and Wu [12] and Ismail [13] . They used the fact that if some of the modes $\boldsymbol{\phi}_r$ are known, then equations (1)–(3) may all be viewed as linear equations for the coefficients in the matrices **K** and **M**. They combined these and other equations; for example, stating that the total mass of the system was known, or that modes were orthogonal with respect to the unknown mass matrix, and solved them in a least squares fashion. In this approach it is easy—in fact, it is an advantage—to assume that only some terms in **K** and **M** are non-zero, while the others are zero.

The present paper is an immediate generalization of the work of Ram and Gladwell [14]. They reconstructed the finite element model of a rod from two mode shapes and one natural frequency. In essence, they took equations (1) for the two modes $\boldsymbol{\phi}_r$ and $\boldsymbol{\phi}_s$ and rearranged them to give a set of linear equations for the terms in the mass and stiffness matrices. As it stands, this method is highly sensitive to errors in the measured data; to overcome this sensitivity they formed an overdetermined set of equations, by using more modal data, and solved them using least squares.

In this paper, we assume that we have some measured modal and natural frequency data which we accept, we have correctly identified the system type, and have constructed a finite element model. We use the fact that the mass and stiffness matrices in a finite element model of a system are constructed by piecing together the corresponding element mass and stiffness matrices, following a recognized assembly procedure. Each element mass and stiffness matrix is constructed from the assumed element shape functions; a typical term in each matrix is the product of an element parameter, i.e., an element mass or stiffness, and an integral over the element. In the identification process, it is the element parameters which are unknown. The integrals are known: they are generally integrals of products of the assumed shape functions or various derivatives of them. We find, as expected, that the straightforward solution of equations (1) for the element parameters is highly sensitive to errors in the modal data. Instead of overcoming this by setting up an overdetermined system using extra modal data, we construct a simple regularization procedure which finds a system near, in a least squares sense, to a given system.

2. RECONSTRUCTING THE FINITE ELEMENT PARAMETERS FROM EIGENDATA

The analytical model for the system is equation (1). If the model is a finite element one, then the mass and stiffness matrices $\mathbf{M}$ and $\mathbf{K}$ are related to the element matrices $\mathbf{M}_i$ and $\mathbf{K}_i$ by

$$\mathbf{M} = \sum_{i=1}^{n} \mathbf{C}_i^T \mathbf{M}_i \mathbf{C}_i, \qquad \mathbf{K} = \sum_{i=1}^{n} \mathbf{C}_i^T \mathbf{K}_i \mathbf{C}_i, \tag{4}$$

where $\mathbf{C}_i$ are connection matrices, and n is the number of elements. The element matrices $\mathbf{M}_i$ and $\mathbf{K}_i$ have the form

$$\mathbf{M}_i = m_i \boldsymbol{\mu}_i, \qquad \mathbf{K}_i = k_i \boldsymbol{\kappa}_i, \tag{5}$$

where m_i and k_i are the physical parameters corresponding to the ith element and $\boldsymbol{\mu}_i$ and $\boldsymbol{\kappa}_i$ are known, square numerical matrices constructed from the assumed shape functions for the ith element. If there is more than one mass and one stiffness parameter corresponding to an element, then the sums in equation (4) can be taken over the number of parameters, rather than over the number of elements.

We now rewrite equation (1), namely

$$\mathbf{K}\boldsymbol{\phi}_r = \omega_r^2 \mathbf{M} \boldsymbol{\phi}_r, \tag{1}$$

where $\mathbf{K}$ and $\mathbf{M}$ have order N, as a set of N equations

$$\mathbf{A}_r \mathbf{k} - \omega_r^2 \mathbf{B}_r \mathbf{m} = \mathbf{0} \tag{6}$$

for the $2n$ unknowns $(k_i, m_i)_1^n$. Each element in $\mathbf{A}_r(\mathbf{B}_r)$ is formed as a sum of terms in $\boldsymbol{\phi}_r$ multiplied by terms in the matrices $\boldsymbol{\kappa}_i(\boldsymbol{\mu}_i)$. To see the gist of the replacement of equation (1) by equation (6), we show a simple example in which $N = 3$ and $\mathbf{K}$ depends on four parameters $(k_i)_1^4$, as shown. Then we have the identity

$$\begin{bmatrix} k_1 & -k_4 & k_2 \\ -k_4 & k_3 & k_4 \\ k_2 & k_4 & k_1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_3 & 0 & -\phi_2 \\ 0 & 0 & \phi_2 & \phi_3 - \phi_1 \\ \phi_3 & \phi_1 & 0 & \phi_2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}.$$

We may carry out this replacement process with $\mathbf{M}\boldsymbol{\phi}_r$ and with $\mathbf{K}\boldsymbol{\phi}_r$ to obtain equation (6). How many equations of the form (6) we need, i.e., how many modes we need to consider to find the $2n$ unknowns $(k_i, m_i)_1^n$ depends on the size of $2n$ relative to N. The equations are homogeneous, so that by themselves they can determine at most the $2n - 1$ ratios of the $2n$ unknowns; to find the k_i and m_i themselves we must have some other information, such as the total mass of the system.

The simplest case is that of a one-dimensional system, e.g., a rod or a beam, fixed at one end and free at the other. For such systems, $n = N$, so that equation (6) for two modes will provide $2n$ equations for the $2n$ unknowns. It was such a system that was considered in reference [14]. We will consider the reconstruction of a cantilever beam model in section 3, comparing our results with those obtained by using the method of Berman and Nagy. We compare with them because they found a model which was consistent with the experimental data; they did not process the data, to make it fit some orthogonality condition, for example.

In general, we must determine the $2n$ parameters by using an overdetermined set of equations. If we do this, we can give added weight to the equations derived from the modes in which we have most confidence, particularly the lower modes.

The remainder of the paper runs as follows. We consider a particular finite element model, for a cantilever beam, in section 3; in section 4 we discuss a simple regularization procedure; in section 5 we apply it to the problem of reconstructing the parameters in the finite element model.

3. A CANTILEVER BEAM MODEL

The most common FEM of a beam is based on Hermitian interpolation with the generalized co-ordinates being the displacements and slopes at the nodes. In order to reconstruct such a model from experimentally acquired modal data we would need to measure slopes at the various points along the beam; this is a particularly difficult experimental procedure. For this reason we construct a FEM based on quadratic spline interpolation with simply displacements as generalized co-ordinates. For the same number of elements, this model is stiffer than the Hermitian one; however, since it has only half the number of degrees of freedom, it gives results as accurate as the other with only a little more computation.

Suppose that the beam $(0, L)$ is divided by knots $(\xi_i)_1^{n+1}$ such that

$$0 = \xi_1 < \xi_2 < \cdots < \xi_{n+1} = L.$$

The quadratic B-spline $S_i(x)$ shown in Figure 1 has support (ξ_{i-1}, ξ_{i+2}) and is given by

$$S_i(x) = \sum_{j=i-1}^{i+2} \frac{\langle x - \xi_j \rangle^2}{\pi'(\xi_j)},$$

where

$$\langle x - \xi_j \rangle^2 = \begin{cases} (x - \xi_j)^2, & x > \xi_j, \\ 0, & x \leqslant \xi_j, \end{cases}$$

and

$$\pi(x) = \frac{-1}{\xi_{i+1} - \xi_i} \prod_{j=i-1}^{i+2} (x - \xi_j).$$

We note that $S_i(x)$ is positive in (ξ_{i-1}, ξ_{i+2}); is zero and has zero slope at ξ_{i-1} and ξ_{i+2}; and has a constant second derivative in each of the intervals (ξ_{i-1}, ξ_i), (ξ_i, ξ_{i+1}),

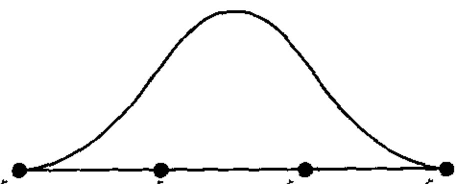

Figure 1. The quadratic B-spline function.

(ξ_{i+1}, ξ_{i+2}), the constants having signs $+$, $-$, $+$. Qualitative properties of these modes and frequencies derived from this model were discussed by Gladwell [15].

The displacement in element i, $\xi_i \leqslant x \leqslant \xi_{i+1}$, is

$$w(x) = \sum_{j=i-1}^{i+1} \alpha_j S_j(x).$$

This means that if the positions of the knots are fixed, then the mass and stiffness parameters for the ith element are

$$m_i = \rho_i A_i, \qquad k_i = E_i I_i,$$

while the coefficients of the matrices $\boldsymbol{\mu}_i$ and $\boldsymbol{\kappa}_i$ correspond to the spline parameters α_i as generalized co-ordinates are

$$(\boldsymbol{\mu}_i)_{jl} = \int_{\xi_i}^{\xi_{i+1}} S_j(x) S_l(x)\,\mathrm{d}x, \qquad (\boldsymbol{\kappa}_i)_{jl} = \int_{\xi_i}^{\xi_{i+1}} S_j''(x) S_l''(x)\,\mathrm{d}x.$$

The eigendata will consist of the values of $w_2, \ldots, w_n$ and ω^2 for the two modes r and s. We may convert the w values to α values by using the relations

$$w_i = \alpha_{i-1} S_{i-1}(\xi_i) + \alpha_i S_i(\xi_i), \qquad \theta_i = \alpha_{i-1} S_{i-1}'(\xi_i) + \alpha_i S_i'(\xi_i),$$

where the second is used with $i = 1$ to satisfy the condition $\theta_1 = 0$ at the fixed end. Having found the α_i for the two modes, we may set up equations (6) for each, i.e.,

$$\mathbf{A}_r \mathbf{k} - \omega_r^2 \mathbf{B}_r \mathbf{m} = 0 = \mathbf{A}_s \mathbf{k} - \omega_s^2 \mathbf{B}_s \mathbf{m}.$$

Since the matrices $\mathbf{K}$ and $\mathbf{M}$ are both pentadiagonal and symmetric, the matrices $\mathbf{A}$ and $\mathbf{B}$ will have a special form; in fact, since the involvement of k_i in the matrix $\mathbf{K}$, and that of m_i in $\mathbf{M}$, is given by

$$\begin{bmatrix} 1 & 2,3 & 3 & & & \\ 2,3 & 2,3,4 & 3,4 & 4 & & \\ 3 & 3,4 & 3,4,5 & 4,5 & 5 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & n-1 & n-1,n & n-1,n & n \\ & & & n & n & n \end{bmatrix},$$

all the matrices $\mathbf{A}_r$, $\mathbf{A}_s$, $\mathbf{B}_r$ and $\mathbf{B}_s$ will be upper-triangular tridiagonal. This means that the last equation of each set (6) will involve k_n and m_n only, i.e.,

$$\begin{bmatrix} a_{n,n}^r - \omega_r^2 b_{nn}^r \\ a_{nn}^s - \omega_s^2 b_{nn}^s \end{bmatrix} \begin{bmatrix} k_n \\ m_n \end{bmatrix} = 0.$$

As in reference [14], this provides a consistency condition relating the two frequencies ω_r and ω_s to the rth and sth modes; namely,

$$\omega_s^2 = \omega_r^2 a_{nn}^s b_{nn}^r / (a_{nn}^r b_{nn}^s). \tag{7}$$

When this condition is satisfied we may find the ratio k_m/m_n from

$$k_n/m_n = \omega_r^2 b_{nn}^r / a_{nn}^r = \omega_s^2 b_{nn}^s / a_{nn}^s,$$

and then solve the remaining equations sequentially for k_{m-i}/m_n, m_{n-i}/m_n, $i = 1, 2, \ldots, n-1$. We can find the absolute values of the stiffness and masses if we know, say, the total mass.

In order to compare our method with that of Berman and Nagy, we performed the following test. We took a datum cantilever beam with $L = 5$, $EI = 1$ and $\rho A = 1$, divided into five equal finite elements. We then changed the mass and stiffness of the second and fourth elements so that the element stiffness and mass parameters were

$$1, 0{\cdot}75, 1, 0{\cdot}75, 1; \qquad 1, 0{\cdot}85, 1, 0{\cdot}85, 1. \tag{8}$$

We computed the first three natural frequencies and mode shapes of this beam using the finite element model. These are shown in Table 1.

We took the second and third computed modes, verified that the consistency condition (7) was satisfied and, not unexpectedly, found that the computed element stiffnesses and masses had precisely the ratios given by equation (8); the mode shapes were computed accurately and the system was so small that there was little or no round-off error. Berman and Nagy's method does not compute the element masses and stiffnesses, but rather finds mass and stiffness *matrices* which are nearest to those of the datum system. Thus, to find the mass matrix **M**, they find **M** nearest to $\mathbf{M}_A$ such that a given modes are orthogonal with respect to **M**; specifically, they use the functional

$$\Psi = \|\mathbf{M}_A^{-1/2}(\mathbf{M} - \mathbf{M}_A)\mathbf{M}_A^{-1/2}\|^2 + \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij}(\boldsymbol{\phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi} - \mathbf{I})_{ij},$$

where $\|\cdot\|$ is the matrix norm given by the square root of the sum of the squares of the matrix elements, and m is the number of modes making up $\boldsymbol{\phi}$. Having found **M**, they find **K** by using the functional

$$\Phi = \|\mathbf{M}^{-1/2}(\mathbf{K} - \mathbf{K}_A)\mathbf{M}^{-1/2}\|^2 + \sum_{i=1}^{N} \sum_{j=1}^{n} \Lambda_{kij}(\mathbf{K}\boldsymbol{\phi} - \mathbf{M}\boldsymbol{\phi}\omega^2)_{ij}$$
$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_{sij}(\mathbf{K} - \mathbf{K}^{\mathrm{T}})_{ij},$$

TABLE 1

First three modes of the cantilever beam

$\omega_1^2 = 0{\cdot}0191$†		$\omega_2^2 = 0{\cdot}8309$		$\omega_3^2 = 7{\cdot}1151$	
Exact	Noisy‡	Exact	Noisy	Exact	Noisy
0·0510§	0·0500	0·2446	0·2420	0·4913	0·5093
0·2000	0·1965	0·6090	0·5953	0·5594	0·5794
0·4164	0·4200	0·5772	0·5635	−0·3938	−0·3798
0·6649	0·6769	−0·0095	−0·0186	−0·4193	−0·4078
0·9262	0·9230	−0·9275	−0·9472	0·9000	0·9159

† Eigenvalue of mode 1.
‡ Mode shape element.
§ Mode shape corrupted with ±5% random errors.

TABLE 2

Identified mass and stiffness matrices using modes 2 and 3 in Table 1

		Identified stiffness						Identified mass $\times 10^3$				
		Berman		Present work				Berman		Present work		
Loc.	Initial stiffness	Exact	Noisy	Exact	Noisy	Noisy regularized	Initial mass $\times 10^3$	Exact	Noisy	Exact	Noisy	Noisy regularized
1,1	6	5·97	5·79	5	463·46	5·03	550	501·1	522·6	482·5	63 540	488·7
1,2	−4	−3·96	−3·97	−3·5	−308·80	−3·53	216·7	195·3	210·9	200·4	−37 480	202·8
1,3	1	0·94	0·84	1·11	4·49	1·01	8·33	26·4	27·86	8·33	208·3	8·41
1,4	0	−0·23	−0·04	0	0	0	0	5·77	−0·911	0	0	0
1,5	0	0·07	0·04	0	0	0	0	−2·04	−3·87	0	0	0
2,2	6	6·29	6·49	−5·5	168·4	−5·44	550	531·3	546·7	535	−27 183	536·6
2,3	−4	−4·03	−4·10	−3·5	−10·03	−3·31	216·7	215·2	218·3	200·4	−2 463	194·6
2,4	1	0·96	1·01	0·75	0·524	0·664	8·33	15·92	9·38	7·08	18·18	65·6
2,5	0	−0·03	−0·08	0	0	0	0	1·97	0·06	0	0	0
3,3	6	−5·73	−5·64	5	7·84	4·56	550	530·8	533·1	482·5	−183·7	454·9
3,4	−4	−3·65	−3·63	−3·5	−3·55	−3·23	216·7	220·1	218·8	200·4	352·9	193·7
3,5	1	0·99	1·01	1	1·25	0·97	8·33	12·63	11·97	8·33	8	8·33
4,4	5	4·90	4·81	4·75	5·52	4·53	500	496·5	499·2	492·5	562·89	489·3
4,5	−2	−2·06	−2·05	−2	−2·5	−1·95	108·3	106·9	107·8	108·3	108·3	108·3
5,5	1	−1·02	1·03	1	1·25	0·97	50	48·89	49·19	50	50	50

where Λ_{kij} and Λ_{sij} are sets of parameters designed to force the satisfaction of the equations of motion and the symmetry of **K**.

In Table 2 are shown the initial stiffness and mass matrices $\mathbf{K}_A$ and $\mathbf{M}_A$ used in Berman and Nagy's method, their computed matrices in the columns marked "exact", and the *truly* exact matrices computed by our method.

We now introduced a random 5% error into the mode shapes, as shown in Table 1, and repeated the reconstructions; the results are shown in Table 2 in the columns marked noisy. Clearly, the noise has little effect on Berman and Nagy's predictions, but dramatically affected ours. We will introduce a regularization in the following section: Berman and Nagy's reconstruction is already regularized.

The natural frequencies and mode shapes computed from Berman and Nagy's models are shown in Table 3. Their model automatically reproduces the modes and frequencies (2 and 3 in this case) that were used to construct it. We note that, whether using the exact or noisy data, the method yields a first mode (Table 3) which is substantially different from the actual first mode (Table 1) of the system from which the modal data (for modes 2 and 3) was constructed. However, this is not the major drawback in Berman and Nagy's method, rather, it is that they predict the **K** and **M** matrices, and not the individual parameters k_i and m_i. Moreover, their matrices do not have the correct connectivity, the correct pattern of zero and non-zero terms, as shown by comparing columns 3 and 4 of Table 2 with column 5, and columns 9 and 10 with column 11.

4. REGULARIZATION

If one of the model parameters, say one of the masses, or the total mass, is known, then the equations used to give the **k** and **m** may be written

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{9}$$

With new definitions of m and n, we may suppose that $\mathbf{A} \in R^{m\times n}$, $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{b} \in R^m$; equations (9) are m equations for n unknowns. Since both **A** and **b** depend on the (noisy) modal and frequency data, this is a classical ill-posed problem. It may be over-, under- or mixed-determined, and small changes in data may lead to large changes in "the solution". To regularize it, we replace it with a constrained least squares problem (Golub and Van Loan [16]):

$$\text{minimize } \|\mathbf{A}\mathbf{x} - \mathbf{b}\|, \qquad \text{subject to } \|\mathbf{B}\mathbf{x} - \mathbf{d}\| \leqslant \alpha.$$

If there is an *a priori* estimate $\mathbf{x}_0$ in which we have more or less confidence, then we may use

$$\mathbf{B} = \text{diag}\,(d_1/x_{1,0}, d_2/x_{2,0}, \ldots, d_n/x_{n,0}). \tag{10}$$

TABLE 3

Modal data of Berman and Nagy's identified model in Table 2

		$\omega_2^2 = 0{\cdot}8309$		$\omega_3^2 = 7{\cdot}1151$	
$\omega_1^2 = 0{\cdot}0286$, exact	$\omega_1^2 = 0{\cdot}0245$, noisy	Exact	Noisy	Exact	Noisy
0·0607	0·0569	0·2446	0·2420	0·4913	0·5093
0·1961	0·1932	0·6090	0·5953	0·5594	0·5794
0·3762	0·3825	0·5772	0·5653	−0·3938	−0·3798
0·6374	0·6414	−0·0095	−0·0186	−0·4193	−0·4078
0·9641	0·9512	−0·9275	−0·9472	0·9000	0·9159

TABLE 4

Identified mass and stiffness parameters of the cantilever beam

Physical parameters	Test modes 1 and 2: Direct solution	Test modes 1 and 2: Reg, $\lambda = 12{\cdot}85$	Test modes 1 and 3: Direct solution	Test modes 1 and 3: Reg, $\lambda = 13{\cdot}42$	Test modes 2 and 3: Direct solution	Test modes 2 and 3: Reg, $\lambda = 0{\cdot}033$	Exact
k_1	0·006	1·145	8·96	1·151	−140·7	0·991	1·000
k_2	0·100	0·901	5·17	0·900	149·9	0·758	0·750
k_3	−0·088	0·791	1·59	0·779	4·492	1·009	1·000
k_4	0·637	0·906	1·38	0·983	0·524	0·645	0·750
k_5	−0·089	0·593	−0·09	0·590	1·250	0·972	1·000
m_1	1193·6	1·000	543·5	1·000	4184·6	0·998	1·000
m_2	−105·4	1·000	−60·57	0·988	−320·9	0·863	0·850
m_3	21·9	0·980	96·1	0·993	−25·00	1·009	1·000
m_4	−5·6	1·023	−3·04	1·018	2·258	0·788	0·850

We take d_r large when we believe that $|x_r - x_{r,0}|$ is small, and *vice versa*.

Clearly, for given α, the constrained problem has a solution if $\min \|\mathbf{Bx} - \mathbf{d}\| \leqslant \alpha$. If this is so, then there are two possibilities: either

$$\|\mathbf{BA}^+\mathbf{b} - \mathbf{d}\| \leqslant \alpha,$$

in which case $\mathbf{x} = \mathbf{A}^+\mathbf{b}$ is the solution, where $\mathbf{A}^+$ is the generalized inverse of $\mathbf{A}$; or $\|\mathbf{BA}^+\mathbf{b} - \mathbf{d}\| > \alpha$ and the solution satisfies the generalized normal equation

$$(\mathbf{A}^T\mathbf{A} + \lambda\mathbf{B}^T\mathbf{B})\mathbf{x} = (\mathbf{A}^T\mathbf{b} + \lambda\mathbf{B}^T\mathbf{d}), \tag{11}$$

where λ is chosen so that

$$\|\mathbf{Bx} - \mathbf{d}\| = \alpha.$$

We obtain the normal equation (11) by using the functional

$$\|\mathbf{Ax} - \mathbf{b}\|^2 + \lambda(\|\mathbf{Bx} - \mathbf{d}\|^2 - \alpha^2). \tag{12}$$

Thus, for a given λ, we call the solution of equation (11), $\mathbf{x}(\lambda)$, put

$$f(\lambda) = \|\mathbf{Bx}(\lambda) - \mathbf{d}\|,$$

and solve the equation

$$f(\lambda) = \alpha^2,$$

TABLE 5

Predicted modal data of the identified cantilever beam model ($\lambda = 0{\cdot}033$)

$\omega_1^2 = 0{\cdot}0193$	$\omega_2^2 = 0{\cdot}7960$	$\omega_3^2 = 6{\cdot}7603$
0·0516	0·2385	0·4856
0·2017	0·6000	0·5635
0·4189	0·5891	−0·4021
0·6702	0·0068	−0·4430
0·9364	−0·9379	0·8856

TABLE 6

Improved solutionn with weighted regularization (modes 2 *and* 3, $\lambda = 0{\cdot}029$)

Element number	Stiffness coefficient		Mass coefficient	
	Computed	Exact	Computed	Exact
1	1·000	1·000	1·000	1·000
2	0·735	0·750	0·845	0·850
3	1·000	1·000	1·000	1·000
4	0·659	0·750	0·857	0·850
5	1·000	1·000	1·000	1·000

using the proposal of Reinsch [17]; namely,

$$\lambda_{j+1} = \lambda_j - \frac{2f(\lambda_j)}{f'(\lambda_j)}\left\{\frac{\sqrt{f(\lambda_j)}}{\alpha} - 1\right\}.$$

An alternative is to use the algorithm proposed by Elaen [18].

5. REGULARIZED RESULTS

We return to the finite element model and use the noisy data shown in Table 1. We used three combinations of the modal data, took $m_5 = 1$ and $(k_{0,i})_1^5 = 1 = (m_{0,i})_1^4$, $\alpha = 0{\cdot}5$ and $(d_i)_1^9 = 1$. The results are shown in Table 4. We used the same value of α for each test and found three different values of λ. The value of λ may be used as a rough estimate of the error in the solution; equation (12) shows that, with a small λ, more weight is given to the first term, the deviation of $\mathbf{Ax}$ from $\mathbf{b}$. The smallest λ was found by using modes 2 and 3; this may be explained intuitively by noting that a 5% error in mode 1, which is monotonically increasing, will have more effect than a corresponding error in either modes 2 or 3. The mass and stiffness matrices computed from the results in Table 4 are shown in the columns marked "noisy regularized" in Table 2. The computed first three frequencies

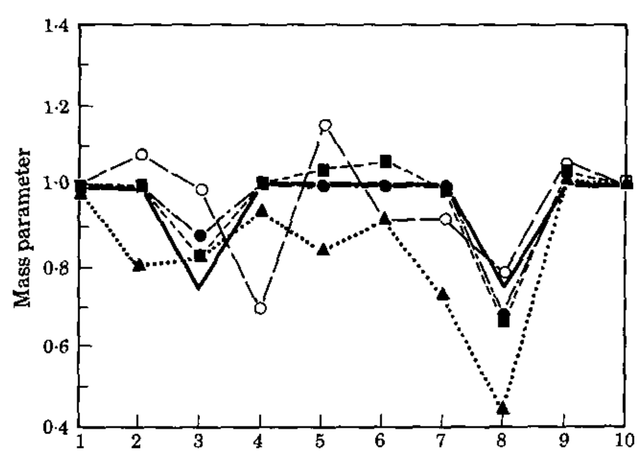

Figure 2. Identified mass parameters of the beam model with ten elements: ——, exact; ○, case 1; ■, case 2; ▲, case 3; ●, case 4.

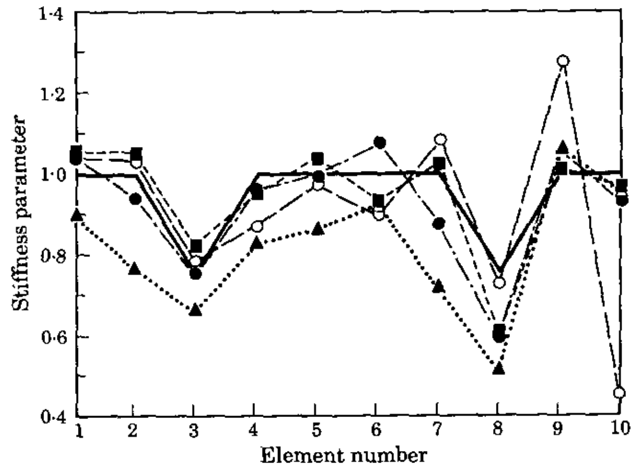

Figure 3. Identified stiffness parameters of the beam model with ten elements: ——, exact; ○, case 1; ■, case 2; ▲, case 3; ●, case 4.

for this model are shown in Table 5; they may be compared with the noisy values shown in Table 1.

Having found a reasonable solution for the physical parameters, by using modes 2 and 3, we may improve the model by fixing the values of those parameters in which we have most confidence, and allowing flexibility in the others. Thus for those terms for which the regularized solution differs from the datum by less than 5%, we take $d_i = 10$: for the others, we take $d_i = 1$. The model so found is shown in Table 6.

Finally, in Figures 2 and 3 and Tables 7 and 8 are summarized the results of a numerical experiment on a cantilever beam modelled with ten elements. In the first line in Table 8 are shown the first five natural frequencies of the beam having the stiffness and mass parameters labelled "exact" in Figures 2 and 3. We considered four cases: in case 1, we took the modal values for modes 2 and 3, corrupted then with 5% error, and assigned equal confidence values to all the masses and stiffnesses ($d_i \equiv 1$); the base values of the masses and stiffnesses were also taken to be all equal. The computed masses and stiffnesses are shown in Figures 2 and 3 and the computed frequencies are shown in line 2 of Table 8. We noted that only two of the computed stiffnesses and two of the computed masses differed significantly from the base values; namely, k_3, k_8 and m_3, m_8. In the second case we assigned less confidence to these values, i.e., we assigned to $d_i = 1$ to them, and $d_i = 10$ to the others. The results of case 2 are "closer" than those of case 1, as indicated by the fact that the λ value is halved. In cases 3 and 4 we repeated the analysis with 3 modes, rather than 2.

TABLE 7

Input data to the identification processes of the beam model with ten elements

Case number	Test modes	α	d_i
1	2 and 3	5·0	$1, i = 1, \ldots, 19$
2	2 and 3	5·0	$1, i = 3, 8$; 10 else
3	2, 3 and 4	2·5	$1, i = 1, \ldots, 19$
4	2, 3 and 4	2·5	$1, i = 3, 8$; 10 else

TABLE 8

Predicted frequencies of the identified beam with ten elements

	λ	ω_1^2	ω_2^2	ω_3^2	ω_4^2	ω_5^2
Exact	—	0·0012	0·0497	0·4000	1·639	4·914
Case 1	0·0110	0·0012	0·0499	0·4044	1·532	4·992
Case 2	0·0050	0·0013	0·0479	0·3838	1·569	4·892
Berman modes 2 and 3	—	0·0015	0·0497	0·4000	1·652	4·897
Case 3	0·0716	0·0011	0·0455	0·3676	1·582	4·773
Case 4	0·0474	0·0013	0·0480	0·3647	1·566	4·756
Berman modes 2, 3 and 4	—	0·0041	0·0479	0·4000	1·539	4·868

It is interesting to note that, in every case, the computed natural frequencies are closer to those used in the reconstruction and, furthermore, the remaining frequencies 1, 4 and 5 in cases 1 and 2, and 1 and 5 in cases 3 and 4, are recovered with good accuracy; this is not so in Berman's analysis using modes 2, 3 and 4; the first natural frequency is not well reproduced.

6. CONCLUSIONS

The reconstruction of a model, finite element or otherwise, from modal and frequency data is an ill-posed problem. Generally, there is no well-defined, unique, solution; instead there is a class of models, each of which predicts approximately the given data; regularization provides a procedure for selecting a particular model from the class.

The regularization which we have used is simply to require that the model which we are seeking be near some preassigned one; in practice, the analyst would have some *a priori* idea of what the model should be.

We have shown that, by rewriting the frequency equations, we may isolate the unknown model parameters, and then apply established regularization procedures to identify an acceptable model.

There are still many matters to be investigated; whether it is better to use higher frequency or lower frequency modes to attempt to construct the model; how the procedure performs in practice when there are equal or very near natural frequencies; how the procedure may be modified to accommodate incomplete mode shapes; how the procedure performs for systems with many degrees of freedom. The answers to these questions are the subject of current research.

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